Aldo Montesano PRINCIPLES OF ECONOMIC ANALYSIS

Chapter 12. GENERAL EQUILIBRIUM ANALYSIS II

12.1 Regular competitive equilibrium

Equilibrium existence is a prerequisite for studying the theory of competitive equilibrium. If the equilibrium did not exist, the theory would contain contradicting relations and would be logically inconsistent. A different logical problem is the uniqueness of the equilibrium, in the sense that the possibility of multiple equilibria existence does not invalidate the logical consistency of the theory (and, possibly, its empirical validity). Nevertheless, it makes the theory unable to determine exchanges, production and prices. Therefore, we need a complementary or more general theory that would allow us to screen among different equilibria by examining the conditions that lead to one or another equilibrium.[1]

Let's consider, as example, the case of an economy with *free disposal*, two goods and a continuous and differentiable aggregate excess demand function that satisfies the desirability condition (according to Definition 11.5, that is $E_1(p_1, p_2) > 0$ for $p_1 = 0$ and $E_2(p_1, p_2) > 0$ for $p_2 = 0$). Therefore, equilibrium prices p_1^*, p_2^* are positive and equilibrium is described by conditions $E_1(p_1^*, p_2^*) = 0$ and $E_2(p_1^*, p_2^*) = 0$. Homogeneity of degree zero of the excess demand function implies that this function has the exchange ratio p_1/p_2 as its argument. Therefore, the prices can be normalized, for example by assuming $(p_1, p_2)\in S^1$, so that $p_1+p_2 = 1$. By desirability condition $E_1(0, 1) > 0$ and $E_2(1, 0) > 0$. As a result of Walras law, since $p_1E_1(p_1, p_2)+p_2E_2(p_1, p_2) = 0$, we get that equilibrium is not only determined by only one condition, for example by $E_1(p_1^*, p_2^*) = 0$ (since the condition $E_2(p_1^*, p_2^*) = 0$, because $p_1^*, p_2^*\in(0, 1)$, is automatically satisfied if the first one is), but also that $E_1(1, 0) = 0$ with $E_1(p_1, p_2) < 0$ for values of p_1 close to 1. (In fact, since by continuity $E_2(p_1, p_2) > 0$ for values of p_2 close to zero, that is for values of p_1 close to 1, Walras law requires $E_1(p_1, p_2) < 0$ for values of p_1 close to 1 and by continuity $E_1(p_1, p_2) \leq 0$ for $p_1 = 1$. Moreover, if $p_1 = 1$, then $E_1 = 0$ because none of the agents wants to

[1] It maybe useful to make an analogy. Let's consider the static equilibrium of a dice. Mechanics makes sure that the rest on a plane is in static equilibrium position if the vertical of the gravity center of the dice crosses the supporting plane in a point belonging to convex hull of the supporting points of the dice. Therefore, a dice has 26 equilibrium positions, from which 6 are stable (the ones in which the dice rests on a side) and 20 are instable (the dice rests on an edge or a vertex). The multiplicity of equilibria does not yield static mechanics, in some sense, false or unimportant, but only insufficient to determine unambiguously the resting position of a dice. (However, keep in mind that the term "static" has a different meaning in mechanics than in economics).

sell his endowment of the first good and so to have a negative excess demand for it as soon as he can have his desired quantity of the second good for free). In Figures 12.1, 12.2 and 12.3 we represent three functions $E_1(p_1, p_2)$ that satisfy all the indicated conditions (that guarantee existence of equilibrium with $p_1^* \in (0, 1)$), but which are different with respect to the number of equilibria. Recall that the desirability condition requires that the prices in equilibrium are positive, that is $p_1^*, p_2^* \in (0, 1)$, and so, even if $E_1(1, 0) = 0$, the prices $p_1 = 1$, $p_2 = 0$ do not describe an equilibrium: in fact $E_2(1, 0) > 0$.

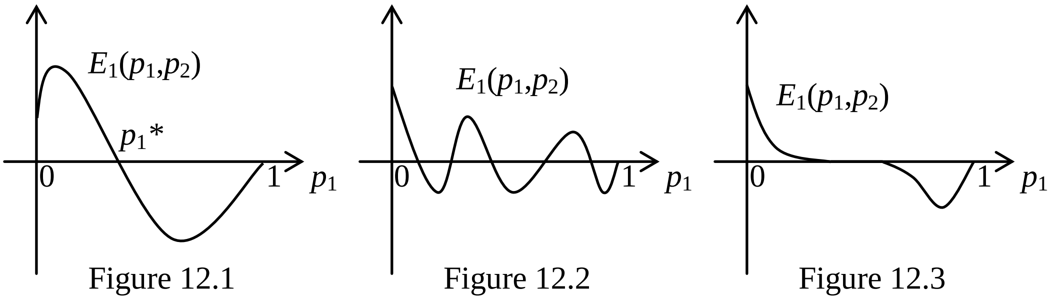

Figure 12.1 Figure 12.2 Figure 12.3

Figure 12.1 depicts the case in which there is only one equilibrium. Figure 12.2 the case in which there is multiplicity of *isolated* (or *locally unique*) *equilibria*. Figure 12.3 the case with a continuum of equilibria (in this figure there is an interval of equilibrium prices).

With respect to an economy with only two goods, these figures allow for an immediate characterization of the conditions that determine which of the cases occurs. We have unique global equilibrium (like in Figure 12.1) if the excess demand function (continuous, differentiable and with both goods desirable) has negative derivative in every equilibrium and only if this derivative is not positive: i.e. if $E_1(p_1^*, p_2^*) = 0$ for $p_1^* < 1$ implies $\mathrm{D}_{p_1}E_1(p_1^*, p_2^*) < 0$ and only if it implies $\mathrm{D}_{p_1}E_1(p_1^*, p_2^*) \leq 0$. We have that an equilibrium is isolated if the excess demand function derivative is not equal to zero in this equilibrium: i.e. (p_1^*, p_2^*) is an isolated equilibrium if $E_1(p_1^*, p_2^*) = 0$, $p_1^* < 1$, and $\mathrm{D}_{p_1}E_1(p_1^*, p_2^*) \neq 0$. All the equilibria are isolated (as in Figure 12.2) if the excess demand function has a non zero derivative in every equilibrium: that is if $E_1(p_1^*, p_2^*) = 0$ for $p_1^* < 1$ implies $\mathrm{D}_{p_1}E_1(p_1^*, p_2^*) \neq 0$. Note that there must be a finite number of isolated equilibria. There is a continuum of equilibria (as in Figure 12.3) only if $\mathrm{D}_{p_1}E_1(p_1^*, p_2^*) = 0$ for a (p_1^*, p_2^*) with $E_1(p_1^*, p_2^*) = 0$.

Finally, note that the situation depicted in Figure 12.3 is not robust with respect to changes in excess demand function. It is enough to change it only a little bit and the continuum of equilibria disappears (as shown in Figure 12.4). The cases depicted in Figures 12.1 and 12.2 are, on the other hand, *robust* (or *structurally stable*). A small change in the excess demand function does not destroy the unique equilibrium (in Figure 12.1) and isolated equilibria and their number (in Figure 12.2), as shown in Figure 12.5. In Figure 12.6 we show, instead, a case in which equilibria are isolated but their number is not robust. Note that, in this figure, the derivative of the excess demand function with respect to price is equal to zero in the non robust equilibrium. Note also that if all equilibria are robust then their number is odd.

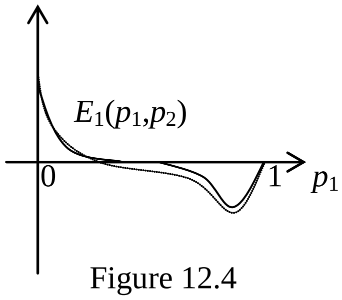

Figure 12.4

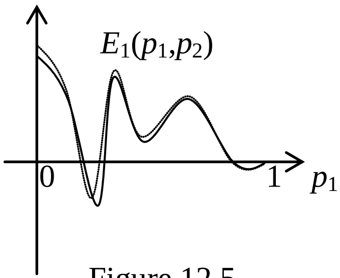

Figure 12.5

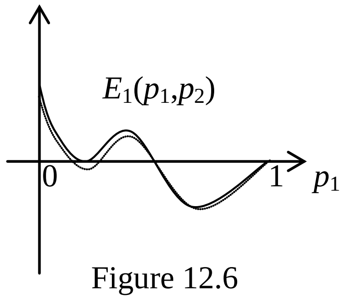

Figure 12.6

We can now conjecture, basing on an economy with two goods, that the robustness property is strictly linked to condition $\mathrm{D}_{p_1}E_1(p_1^*, p_2^*) \neq 0$. So far we have shown that it holds for an economy with two goods. Now we need to verify whether it holds for k goods.

In the following analysis we study economies with single-valued[2] aggregate excess demand functions (that is, not correspondences but proper functions) $E\colon \mathbb{R}^k_+ \to Z$, where $Z = \sum_{i=1}^n X_i - \sum_{j=1}^m Y_j - \{\Omega\}$, assuming that they are continuous, differentiable, homogenous of degree zero (that is, $E(\alpha p) = E(p)$ for every $\alpha > 0$ (so the prices can be normalized for example setting $p \in S^{k-1}$) and satisfy Walras law (that is, $pE(p) = 0$ for every $p \in S^{k-1}$) and desirability condition for all the goods (that is, for every $h = 1,\ldots, k$, we have $E_h(p) > 0$ for every $p \in S^{k-1}$ with $p_h = 0$).

It has already been noted that for such economies equilibrium exists and that every equilibrium vector of prices is positive (Paragraphs 11.4 and 11.6). Equilibrium vectors of prices p^* in such economies are determined by the following equations

$$E(p) = 0$$

This system has not more than k–1 independent equations because of the linear dependence introduced by Walras law. There are also k–1 unknowns even if there are k prices, since functions $E(p)$ are homogenous of degree

[2] In production economy this hypothesis excludes possibility of constant returns to scale. As a result, very often the following arguments are referred to as to pure exchange economies. There are, however, extensions that apply also to the cases in which aggregate excess demand is a correspondence. Dierker (1982) presents a review of the questions that we rise in this paragraph including also this case in the analysis.

zero, so that we can normalize prices, by for example setting $p \in S^{k-1}$, so with $\sum_{h=1}^{k} p_h = 1$. We can discard one equation from the system, for example the k-th equation (it will be automatically satisfied if the others are) and one price, for example p_k (it is automatically determined when other prices are determined). Therefore, we can consider excess demands for the first k–1 goods and their prices, that is vectors $\tilde{E} = [I \vdots 0]E$ and $\tilde{p} = [I \vdots 0]p$, where I is the identity matrix with k–1 rows and columns and 0 is the zero vector. Thus, in what follows, we consider a system of k–1 equations

$$\tilde{E}_h(p_1,\ldots, p_k) = 0, \qquad h = 1,\ldots, k-1$$

and a normalization condition, for example $p \in S^{k-1}$, when we want to determine all the prices (not only the exchange ratios).

The following propositions are independent of the normalization rule that we adopted for the prices (even if they are presented using the normalization $p \in S^{k-1}$). To be precise, we could obtain the same properties independent of the kind of price normalization applied. Another, frequently used, normalization (that is possible because the prices in equilibria are positive) sets the price of one of the goods equal to one, for example $p_k = 1$. That is, it takes the k-th good as the accounting unit, and so the prices become the exchange ratios with respect to this good.

Let's introduce the following definitions

Definition 12.1 (*Regular equilibrium and economy*) An equilibrium vector of prices $p^* \in S^{k-1}$ is regular if the Jacobian matrix $\mathrm{D}_{\tilde{p}}\tilde{E}(p^*)$ is not singular (i.e. its determinant is not equal to zero). An economy is regular if all the equilibrium price vectors are regular.

The regularity condition implies local uniqueness and a finite number of equilibria, as shown in the following propositions.

Proposition 12.1 In a regular equilibrium every vector of prices is isolated (or locally unique), that is if $p^* \in S^{k-1}$ is regular, then there exists an $\varepsilon > 0$ such that $\tilde{E}(p) \neq 0$ for every $\tilde{p} \neq \tilde{p}^*$ with $\| \tilde{p} - \tilde{p}^* \| < \varepsilon$, where $\tilde{p} = [I \vdots 0]p$ with $p \in S^{k-1}$.

Proof. Taylor series expansion of $\tilde{E}(p)$ in the neighborhood of $p^* \in S^{k-1}$, since $\tilde{E}(p^*) = 0$, gives $\tilde{E}(p) = \mathrm{D}_{\tilde{p}}\tilde{E}(p^*)\,\mathrm{d}\tilde{p} + \ldots$ As a result, we get $\tilde{E}(p) \neq 0$ for every $\mathrm{d}\tilde{p} \neq 0$ because the matrix $\mathrm{D}_{\tilde{p}}\tilde{E}(p^*)$ is not singular. By continuity, we get $\tilde{E}(p) \neq 0$ in the neighborhood of p^* in S^{k-1}. □

Proposition 12.2 A regular economy has a finite number of equilibrium price vectors.

Proof. By continuity of function $\tilde{E}(p)$ we get that the set of equilibrium price vectors (derived from condition $\tilde{E}(p) = 0$) is closed. This set is also bounded (since S^{k-1} is bounded). Moreover, it is discrete, because its points, by Proposition 12.1, are isolated. As a result, the set of equilibrium price vectors, that is compact and discrete, is also finite and composed of a finite number of points. □

We can also prove that a regular economy has an odd number of equilibria. In order to do it, we apply a differential topology theorem (*index theorem*).

Definition 12.2 (*Index of a regular equilibrium and of a regular economy*) We define

$$\text{ind } p^* = (-1)^{k-1} \text{ sign det } \mathrm{D}_{\tilde{p}}\tilde{E}(p^*)$$

as the index of a regular equilibrium price vector, where the symbol "sign" is such that sign $\beta = 1$ if $\beta > 0$ and sign $\beta = -1$ if $\beta < 0$. Then, recalling that by Definition 12.1 det $\mathrm{D}_{\tilde{p}}\tilde{E}(p^*) \neq 0$, the index can take on value 1 or -1. (In case with only two goods, the index is equal to 1 if $\mathrm{D}_{p_1}E_1(p_1^*, p_2^*) < 0$ and equal to -1 if $\mathrm{D}_{p_1}E_1(p_1^*, p_2^*) > 0$). We define the index of a regular economy, recalling that it has a finite number of equilibria, as the sum of the indexes of its equilibrium price vectors, that is $\sum_{\{p^* \in S^{k-1}: \tilde{E}(p^*)=0\}} \text{ind } p^*$.

Proposition 12.3 If an economy is regular, then $\sum_{\{p^* \in S^{k-1}: \tilde{E}(p^*)=0\}} \text{ind } p^* = 1$. This implies that the number of equilibrium price vectors is odd. In fact, a sum of elements equal to 1 or to -1 is equal to 1 only if the number of the elements is odd.

Proof. Instead of the formal proof we provide an intuition for the fact that a regular economy has an odd number of equilibria. Let's introduce an economy with the same goods as the economy under examination and with only one equilibrium price vector. Moreover, let this economy be characterized by a continuous and differentiable excess demand function $\hat{E}(p)$ and by a positive equilibrium price vector $\hat{p}^* >> 0$. Consider the function $E'(p,\lambda) = \lambda E(p) + (1-\lambda)\hat{E}(p)$ with $\lambda \in [0, 1]$, where $E(p)$ is the aggregate excess demand function of the examined economy, and consider the equilibrium condition $E'(p,\lambda) = 0$. This system is composed of k–1 equations and k unknowns (k–1 prices and λ). The solutions, which are the points in the set $\{p \in S^{k-1}, \lambda \in [0, 1]: E'(p,\lambda) = 0\}$, have one degree of freedom and are represented by curves in the set $\{p \in S^{k-1}, \lambda \in [0, 1]\}$. These prices are never equal to zero (or to 1) because desirability condition holds. In Figure 12.7 we show a possible situation for the case with $k = 2$ with respect to the price p_1 (recall that $p_2 = 1 - p_1$), The same reasoning can be used for the case with $k > 2$. The solutions $\{p \in S^1, \lambda \in [0, 1]: E'(p,\lambda) = 0\}$ are depicted by the curves drawn inside the rectangle. They are either closed curves or they intersect the sides of the rectangle defined by $\lambda = 0$ and $\lambda = 1$. The curves never break for $\lambda \in (0, 1)$ because function $E'(p,\lambda)$ is continuous. (In fact, since the sign of $E'(p,\lambda)$ is different over and under every part of the curve, while on the curve we have $E'(p,\lambda) = 0$, if the curve breaks for $\lambda \in (0, 1)$, then we can go from a negative value of $E'(p,\lambda)$ to a positive value without crossing at zero, a possibility that is excluded by

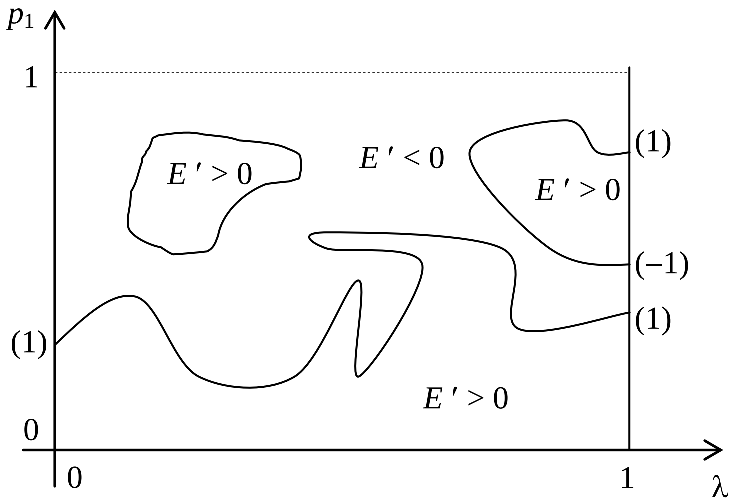

Figure 12.7

the continuity of $E'(p,\lambda)$). Moreover, since the examined economy is regular, there are no curves tangent to $\lambda = 1$. There is a unique intersection at $\lambda = 0$, because $\hat{E}(p) = 0$ has a unique solution. Then, since there is only one intersection point at $\lambda = 0$, as every curve has an even number or none intersections with the sides of the rectangular defined by $\lambda = 0$ and $\lambda = 1$, it follows that there is an odd number of the intersection points at $\lambda = 1$. Therefore, $E(p) = 0$ has an odd number of solutions. (The indexes for the solutions at $\lambda = 0$ and $\lambda = 1$ are shown in the figure in parenthesis and are determined by observing the changes of the sign of E'). □

Until now we have considered regular economies. Now, we will examine if they generally occur or not, in order to understand whether non regular economies could be relevant as well. (This problem is connected to something that we have already seen for an economy with two goods, where the non regular economies were shown to be structurally unstable). We will show, in the following reasoning, that regularity is a generic (or normal) property. In order to do it we will first define what a *generic property* is.

Consider a set of economies and the set of the aggregate excess demand function $E(p)$ corresponding to them. We can represent this set of functions (one for each economy) with $\{E(p;c) : c \in C\}$, where C is a set of parameters, such that each of its elements c characterizes one economy. That is, each economy is characterized by one particular vector c, so its aggregate excess demand function becomes $E(p;c)$. In rough terms, a property of $E(p)$ is generic if it holds for every c in C or for almost every c in C. We will soon provide a more precise definition for a case in which *C* is a subset of an Euclidian space with a finite number of dimensions γ that define the measure. (If *C* is an interval in $\mathbb{R}$, that is γ = 1, then the possible measure is length; if *C* is a non degenerate subset of $\mathbb{R}^2$, that is γ = 2, the measure could be an area; and so on).[3]

A definition of generic property (analysis studied and presented by Mas–Colell, 1985) states that a property is generic if it holds for a subset of *C* of full measure, that is it does not hold in a subset of *C* of zero measure. (If *C* is an interval in $\mathbb{R}$, then a property is generic if it holds for every *c* in *C*, except, possibly, in isolated points of *C*, etc.).

In order to understand the significance of a generic property one should realize that if *c* is determined according to a non atomic probability distribution over *C* (as is the uniform distribution, or the normal non degenerate distribution), then the probability that the generic property does not hold is equal to zero (that is, the generic property occurs with probability equal to 1, although not necessarily with certainty). Another intuitive explanation (close to the notion of structural stability) indicates that a property is generic if it is robust to a perturbation of *c* (that is, it holds for almost all points in the neighborhood of *c*).

The proposition that states that the regularity of an economy is a generic property comes out of the *transversality theorem* (that will be presented without a proof, with respect to the examined problem, considering, as usual, $\tilde{E} = [I \vdots 0]E$ and $\tilde{p} = [I \vdots 0]p$).

Transversality theorem. If function $E(p;c)$, with $E : S^{k-1} \times C \rightarrow Z$, has, with respect to the pairs (p,c) for which $E(p;c)$ = 0, Jacobian matrix $\mathrm{D}_{\tilde{p}.c}\tilde{E}(p;c)$ with rank *k*–1, then the submatrix $\mathrm{D}_{\tilde{p}}\tilde{E}(p;c)$ has, with respect to the pairs (p,c) for which $E(p;c)$ = 0, generically (that is, for almost every *c*) rank *k*–1.

Note that the matrix $\mathrm{D}_{\tilde{p}.c}\tilde{E}(p;c)$ has *k*–1 rows and *k*–1+γ columns, while the matrix $\mathrm{D}_{\tilde{p}}\tilde{E}(p;c)$ has *k*–1 rows and columns, so the condition that the rank is equal to *k*–1 is much more stringent for the second matrix. This theorem implies that if the economies are sufficiently variegated, then regularity is a generic property, that is it is almost certain (with probability equal to 1 for a randomly chosen economy) that the regularity condition is satisfied.

In Paragraph 12.5, where we analyze comparative statics of general competitive equilibrium, we consider, in particular, a case in which the set

[3] There exists a general theory of measure which one can refer to for extensions (presented, for example, by Kirman, 1981). A measure can also be defined for functional spaces (i.e., with infinite dimensions). Moreover, the definition of generic property can be provided even when measurability is absent.

of the possible economies consists of the economies that differ from each other in the endowments of the consumers. Therefore, the characteristic c is represented by the initial allocation $\omega \in \mathbb{R}^{nk}_{+}$, where $\omega = (\omega_{11},\ldots, \omega_{1k},\ldots, \omega_{n1},\ldots, \omega_{nk})$. The following proposition shows that the matrix $\mathrm{D}_{\omega}\tilde{E}(p;\omega)$ and, consequently, the matrix $\mathrm{D}_{\tilde{p},\omega}\tilde{E}(p;\omega)$ are of rank k–1.

Proposition 12.4 The matrix $\mathrm{D}_{\omega}\tilde{E}(p;\omega)$ is of rank k–1 for every $p\in S^{k-1}$ and $\omega \in \mathbb{R}^{nk}_{+}$ with $\Sigma_{i=1}^{n}\omega_i \in \mathbb{R}^{k}_{++}$.

Proof. If we prove that the rank of the Jacobian matrix $\mathrm{D}_{\omega}\tilde{E}(p;\omega)$ is equal to k–1 for one particular change of ω, then we have also proved that $\mathrm{D}_{\omega}\tilde{E}(p;\omega)$ is of rank k–1 with respect to the general change of ω. Let's order the consumers in such a way that $\omega_{1k} > 0$ and let's consider a change $\mathrm{d}\omega_1$ of the first consumer's endowment that leaves his wealth unchanged, that is with $\sum_{h=1}^{k-1} p_h \mathrm{d}\omega_{1h} + (1-\sum_{h=1}^{k-1} p_h)\mathrm{d}\omega_{1k} = 0$. The bundle of goods that he chooses is not modified by this change. His excess demand that was $e_1(p;\omega_1)$ before the change becomes $e_1(p;\omega_1) - \mathrm{d}\omega_1$, so the change of his excess demand is equal to $-\mathrm{d}\omega_1$. Since nothing changes for the other agents, this is also the change of aggregate excess demand. The change $(\mathrm{d}\omega_{1h})_{h=1}^{k-1}$ is arbitrary, because $\omega_{1k} > 0$ and $\mathrm{d}\omega_{1k} = -\dfrac{1}{p_k}\sum_{h=1}^{k-1} p_h \mathrm{d}\omega_{1h}$ (recalling that $p_k\in(0, 1)$). Denoting the vector $(\mathrm{d}\omega_{1h})_{h=1}^{k-1}$ with $\mathrm{d}\tilde{\omega}_1$, we get that the excess demand for the first k–1 goods, that was $\tilde{E}(p;\omega)$ before the change, becomes $\tilde{E}(p;\omega) - \mathrm{d}\tilde{\omega}_1$, so $\mathrm{D}_{\tilde{\omega}_1}\tilde{E}(p;\omega) = -I$, where I is the identity matrix of rank k–1, and symbol $\mathrm{D}_{\tilde{\omega}_1}$ denotes the matrix of the derivatives with respect to $(\omega_{1h})_{h=1}^{k-1}$. □

Applying the transversality theorem to Proposition 12.4, we immediately get the following proposition.

Proposition 12.5 The regularity of an economy represented by an aggregate excess demand function $E(p;\omega)$ is a generic property in the space of the economies characterized by $\omega \in \mathbb{R}^{nk}_{++}$.

12.2 Global uniqueness of competitive equilibrium

The condition that the economy is regular together with index theorem allow us to have locally unique equilibria. Moreover, if we impose one more condition we are able to have also global uniqueness. This condition is an extension of what we have already seen for an economy with only two goods. Namely, a regular economy, represented by a continuous and differentiable excess demand function that satisfies desirability condition, exhibits only one equilibrium if and only if the equilibrium implies that the derivative of the excess demand function is negative. Recall that $\tilde{E} = [I \vdots 0]E$ and $\tilde{p} = [I \vdots 0]p$.

Proposition 12.6 A regular economy, characterized by a continuous and differentiable excess demand function $E\colon S^{k-1} \to Z$ that satisfies desirability condition for all the goods, has only one equilibrium if and only if the equilibrium condition $E(p^*) = 0$ implies $(-1)^{k-1}\det \mathrm{D}_{\tilde{p}}\tilde{E}(p^*) > 0$, i.e. $\det(-\mathrm{D}_{\tilde{p}}\tilde{E}(p^*)) > 0$.

Proof. On one hand, since the economy is regular, the sum of the indexes (each equal to 1 or −1) of equilibrium price vectors is equal to 1 by Proposition 12.3. On the other hand, the condition "$E(p^*) = 0$ implies $(-1)^{k-1}\det \mathrm{D}_{\tilde{p}}\tilde{E}(p^*) > 0$" requires, recalling Definition 12.2, that the index is equal to 1 in every equilibrium. Consequently, the proposed condition is both necessary and sufficient for the global uniqueness of equilibrium. □

Proposition 12.6 is very formal. It does not highlight the characteristics of the economy with only one equilibrium. Nevertheless, we can draw the following, economically significant, implication.

If the Jacobian matrix $\mathrm{D}_pE(p)$ of the aggregate excess demand function $E(p)$ is symmetric and negative semidefinite, then $(-1)^{k-1}\det \mathrm{D}_{\tilde{p}}\tilde{E}(p) \geq 0$. (In fact its principal minors of even order have non negative sign and the odd ones have non positive sign. Therefore, $\det \mathrm{D}_{\tilde{p}}\tilde{E}(p)$, which is a principal minor of order $k-1$, has non negative sign if k is odd and non positive sign if k is even). Recalling that $E(p) = \sum_{i=1}^{n} d_i(p) - \omega_i - \sum_{j=1}^{m} s_j(p)$ and that the sum of (symmetric) negative semidefinite matrices is a (symmetric) negative semidefinite matrix, the condition $(-1)^{k-1}\det \mathrm{D}_{\tilde{p}}\tilde{E}(p) \geq 0$ is satisfied if all Jacobian matrices $\mathrm{D}_pd_i(p)$, for $i = 1,\ldots, n$, and $-\mathrm{D}_ps_j(p)$, for $j = 1,\ldots, m$, are symmetric and negative semidefinite. The analysis of production choice guarantees that the matrices $\mathrm{D}_ps_j(p)$ are symmetric and positive semidefinite (Proposition 5.8), so the matrices $-\mathrm{D}_ps_j(p)$ are symmetric and negative semidefinite. However, the analysis of consumption choice (in particular, Paragraph 4.5 and Proposition 3.14) indicates that the matrices $\mathrm{D}_pd_i(p)$ are composed of two parts, the one that shows the substitution effect corresponds to Slutsky matrix which is symmetric and negative semidefinite, while the other, that shows the income effect, generally is neither symmetric nor negative semidefinite and so the matrices $\mathrm{D}_pd_i(p)$ are not in general negative semidefinite. They are negative semidefinite if the substitution effect prevails, that is if the income effect is small with respect to substitution effect. We can, therefore, deduct that equilibrium is globally unique if consumers' demand functions show an income effect that is much smaller than substitution effect, or if income effect does not change the sign of $\det \mathrm{D}_{\tilde{p}}\tilde{E}(p)$. The following definitions and propositions are logically linked to this result.

Definition 12.3 (*Gross substitutes goods for the aggregate excess demand function*) The h-th good is gross substitute with respect to the t-th good, with $h \neq t$ and $h, t = 1,\ldots, k$, if $\dfrac{\partial E_h(p)}{\partial p_t} > 0$.

Proposition 12.7 A regular economy, characterized by a continuous and differentiable aggregate excess function $E: S^{k-1} \to Z$ that satisfies desirability condition for all the goods, has only one equilibrium if all the goods are gross substitutes for each other in equilibrium. That is, for every $p^* \in S^{k-1}$ with $E(p^*) = 0$, we have $\mathrm{D}_{p_t} E_h(p^*) > 0$ for every $h, t = 1,\ldots, k$ with $h \neq t$.

Proof. By Walras law $pE(p) = 0$ for every $p \in S^{k-1}$. Differentiating this relationship with respect to p, we obtain $(\mathrm{D}_p E(p))^{\mathrm{T}} p + E(p) = 0$. So, we get that in every equilibrium $(\mathrm{D}_p E(p^*))^{\mathrm{T}} p^* = 0$. Let's consider the excess demand functions for the first k–1 goods and their prices, that is the vectors $\tilde{E} = [I \vdots 0]E$ and $\tilde{p} = [I \vdots 0]p$, where I is the identity matrix with k–1 rows and columns and 0 is the zero vector. Recalling that the gross substitution condition requires $\mathrm{D}_{p_h} E_k(p^*) > 0$ for every $h \neq k$ and that $p_k^* > 0$, we get $(\mathrm{D}_{\tilde{p}}\tilde{E}(p^*))^{\mathrm{T}}\, \tilde{p}^* << 0$, i.e. $(-\mathrm{D}_{\tilde{p}}\tilde{E}(p^*))^{\mathrm{T}}\, \tilde{p}^* >> 0$. The matrix $(-\mathrm{D}_{\tilde{p}}\tilde{E}(p^*))^{\mathrm{T}}$ has, by gross substitution, all the elements negative, except for those on the main diagonal. Therefore, according to Hawkins-Simon[4] conditions, all the principal minors of the matrix $(-\mathrm{D}_{\tilde{p}}\tilde{E}(p^*))^{\mathrm{T}}$, and so also the ones of the matrix $-\mathrm{D}_{\tilde{p}}\tilde{E}(p^*)$, are positive. Thus, the determinant of the matrix $-\mathrm{D}_p \tilde{E}(p^*)$ is positive, that is $\det(-\mathrm{D}_{\tilde{p}}\tilde{E}(p^*)) = (-1)^{k-1} \det \mathrm{D}_{\tilde{p}}\tilde{E}(p^*) > 0$. As a result, by Proposition 12.6, the equilibrium is unique. □

The condition that all the goods are gross substitutes is strong, in particular in case of production. (In fact, an increase in price of one input can reduce the demand for the other input used in the same production even if the inputs are not complements, because it reduces the quantity of output).

Another very severe sufficient condition for equilibrium uniqueness is obtained by extending the weak axiom of revealed preferences (*WARP*) to aggregate excess demand function. *WARP* (Definition 4.1) is always satisfied by the demand function of one consumer (represented by a regular system of preferences), but not by the aggregate demand function (as shown

[4] According to Hawkins-Simon conditions (introduced in the input-output analysis presented in footnote 15, Chapter 5), for a square matrix B with all components non positive, except for those on the main diagonal, i.e. with $b_{ij} \leq 0$ for $i \neq j$, the followings are equivalent: *a*) there exists an $x \geq 0$ such that $Bx > 0$; *b*) all the principal minors of B are positive.

and presented in Proposition 4.9). For one consumer *WARP* requires that if $pe_i(p') \le 0$ and $e_i(p') \ne e_i(p)$, then $p'e_i(p) > 0$ (by Definition 4.1, setting $m_i = p\omega_i$, $m_i' = p'\omega_i$, $e_i(p) = d_i(p,m)-\omega_i$ and $e_i(p') = d_i(p',m')-\omega_i$). For the aggregate excess demand function *WARP* would require that if $pE(p') \le 0$ and $E(p') \ne E(p)$, then $p'E(p) > 0$. The following proposition holds.

Proposition 12.8 A regular economy, characterized by an aggregate excess demand function $E: S^{k-1} \to Z$ that satisfies desirability condition for all the goods, has only one equilibrium if the aggregate excess demand function satisfies the weak axiom of revealed preferences, that is if the inequalities $pE(p') \le 0$ and $E(p') \ne E(p)$ imply $p'E(p) > 0$.

Proof. By contradiction, let there be two equilibrium price vectors $p^*,p^{**} \in S^{k-1}$ and let $p = \alpha p^*+(1-\alpha)p^{**}$ with $\alpha \in (0, 1)$. If it were $E(p) \ne 0$, then, since $E(p^*) = E(p^{**}) = 0$ and so $pE(p^*) = pE(p^{**}) \le 0$, by *WARP* we get that $p^*E(p) > 0$ and $p^{**}E(p) > 0$ and so also $(\alpha p^*+(1-\alpha)p^{**})E(p) > 0$, that is $pE(p) > 0$. Then, since Walras law requires $pE(p) = 0$, we must have $E(p) = 0$ for every $p = \alpha p^*+(1-\alpha)p^{**}$, which is in contradiction with the condition imposed by regularity of the economy, that says that equilibria are isolated. Therefore, there cannot exist two equilibrium price vectors. □

Weak axiom of revealed preferences, as already mentioned, is not in general satisfied by the aggregate excess demand function. However, it is satisfied in some particular cases. For example, *WARP* is satisfied if the Antonelli-Nataf-Gorman aggregation conditions (introduced in Proposition 4.6) hold. In fact, in such a case, by Proposition 4.7, aggregate demand function of consumers is rationalizable. That is, it is possible to introduce a representative agent whose demand function $D(p,M)$, where $M = p\Omega+pS(p)$, coincides with the aggregate demand function. The wealth of the representative agent is equal to the aggregate wealth of the consumers, that is, for $i = 1,\dots, n$, $m_i = p\omega_i+\sum_{j=1}^{m}\theta_{ij}ps_j(p)$, by which $M = \sum_{i=1}^{n} m_i = p\Omega + p\sum_{j=1}^{m} s_j(p) = p\Omega + pS(p)$. The function $D(p, M)$ satisfies *WARP*, so, by Definition 4.1, we get that $pD(p',M') \le M$ and $D(p',M') \ne D(p,M)$ imply $p'D(p,M) > M'$. Since $E(p) = D(p,M)-\Omega-S(p)$, we get that $p(E(p')+\Omega+S(p')) \le p(\Omega+S(p))$ implies $p'(E(p)+\Omega+S(p)) > p'(\Omega+S(p'))$. That is, $pE(p') \le pS(p)-pS(p')$ implies $p'E(p) > p'S(p')-p'S(p)$, if $E(p')+S(p') \ne E(p)+S(p)$. In the examined case, where supply is a single-value function, the profit maximization (that holds in aggregate as shown in Paragraph 5.7) requires not only $pS(p)-pS(p') \ge 0$ and $p'S(p')-p'S(p) \ge 0$, but also $pS(p)-pS(p') > 0$ and $p'S(p')-p'S(p) > 0$, since otherwise both $S(p)$ and $S(p')$ would maximize profit. As a result, condition $pE(p') \le 0$ implies $pE(p') \le pS(p)-pS(p')$ and this implies $p'E(p) > p'S(p')-p'S(p) > 0$. That is, as required by Proposition 12.8, $pE(p') \le 0$ implies $p'E(p) > 0$ if $E(p') \ne E(p)$. (Note, moreover, that the conditions $pE(p') \le 0$ and $E(p') \ne E(p)$ imply $E(p')+S(p') \ne E(p)+S(p)$, since, otherwise, that is if $E(p')+S(p') = E(p)+S(p)$, we would get $pE(p') = pE(p)+pS(p)-pS(p') = pS(p)-pS(p') > 0$, while by assumption $pE(p') \le 0$).

12.3 Equilibrium stability in logical time (or with *tâtonnements*)

The problem of equilibrium stability arises in the case of general competitive equilibrium, just like for the partial equilibrium before. The purpose of this analysis is to characterize equilibria with respect to their possible occurrence. Therefore, by testing whether the market behavior leads to them or not. Note that analysis of equilibrium stability is concerned with the phase in which equilibrium is established (indicated in Chapter 10). It consists of negotiation and contracting process that gives rise to equilibrium prices and allocations. (The analysis that we have carried out until now has, on the contrary, examined the conditions that define equilibrium but not the process that leads to it). Therefore, equilibrium is defined as stable if it is a result of negotiation and contracting. A way to test equilibrium stability is to perturb it and observe whether after perturbation stops equilibrium reemerges.

An analogy to mechanics can be useful. Imagine a material point (that is, a point with a mass) resting on the ground and under gravity force. Stationary equilibria are the points of the ground with zero slope, as shown in Figure 12.8. Some of those points are stable equilibria (points S_1, S_2 and S_3), some unstable (points I_1, I_2 and I_3). A small perturbation from the first ones (represented as a change of the position of the material point) is followed, after the perturbation stops, by a movement towards the previous equilibrium position. The perturbation from the second ones is followed by a movement away from them.

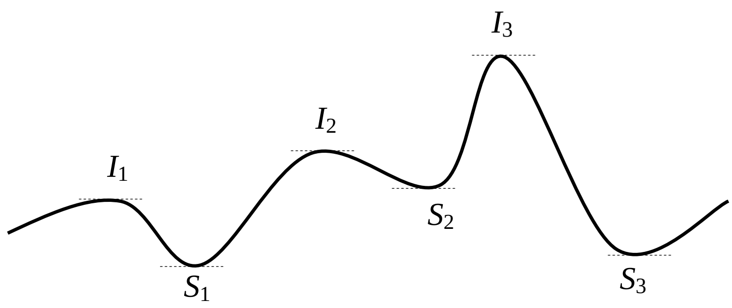

Figure 12.8

From this figure we can deduce that determination of equilibrium and stability conditions (recalling, respectively, that the drawn profile has first derivative equal to zero and, in equilibrium position, the second one positive) does not require complete determination of the material point dynamics, but only of one of its components (that is only that the gravity force is directed downward). In other words, stability property concerns equilibrium position, also if the notion of stability presupposes a dynamic adjustment process, that is not to be necessarily specified, in the sense that the same position can represent a stable equilibrium with respect to multiple dynamic processes. (In Paragraph 10.4 we show how partial equilibrium is stable if the demand function is decreasing and supply function is increasing both for the dynamic Walrasian process and the Marshallian process).

After an equilibrium is defined, in order to research its stability conditions we need to form a hypothesis how to perturb it and what dynamic process will occur afterwards. Stability conditions are those that determine return to equilibrium. In what follows, we hypothesize that the perturbation hits the prices and that the dynamic process is realized by a Walrasian auctioneer, introduced in Paragraph 10.4. The auctioneer announces a vector of prices (note that the market is unique also if there are k goods), to which the agents (consumers and producers) respond by reporting their willingness to buy and sell. The reported values are executed only if the price vector constitutes an equilibrium. The auctioneer controls whether there is an equilibrium, that is whether the aggregate excess demand is zero (or not positive, if we allowed for free goods). If it is zero, then the auctioneer closes the process and the agents execute the exchanges that they communicated to the auctioneer. If the aggregate excess demand is not equal to zero, the auctioneer announces a new vector of prices, and so on. During the dynamic process represented by auctioneer's sequence of price announcements no exchanges or production processes are carried out and other "data" about this economy (number of agents, their preferences and endowments, number of firms and their technology) remain unchanged. In this sense, the contracting phase takes place in a fictitious time, logical time, when the examined economy does not undergo any modification. (There are other types of stability analysis, which we examine in the following paragraph, that allow for exchanges during contracting phase and so the economy is modified. In such a case, the adjustment to equilibrium takes place in *real time*). The analysis in logical time is substantially based on an assumption about the duration of the adjustment process of market prices: it is supposed to be much shorter than the time required by changes in the "data" of the economy. The dynamic process is determined by auctioneer's announcements. Following the Walrasian assumptions, we assume that auctioneer proceeds by *tâtonnements*, rising prices of the goods that have positive excess demand and reducing prices of the goods with negative excess demand. This rule is often called the *market law*. In the following analysis we assume that this dynamic process occurs continuously, that is that a sequence of prices announced by the auctioneer is represented by function $p(t)$, with $t \in \mathbb{R}_+$.

For an economy with only two goods we can refer to Figure 10.5. Let aggregate excess demand function for the two goods be $E_1(p_1/p_2)$ and $E_2(p_1/p_2)$. In such a case, an equilibrium (x^*, y^*, p_1^*, p_2^*), where $x^* = (x_i^*)_{i=1}^{n}$ and $y^* = (y_j^*)_{j=1}^{m}$, is stable if $\mathrm{D}_{p_1/p_2}E_1(p_1^*/p_2^*) < 0$ (and so, accounting for Walras law, $\mathrm{D}_{p_1/p_2}E_2(p_1^*/p_2^*) > 0$), and it is unstable if $\mathrm{D}_{p_1/p_2}E_1(p_1^*/p_2^*) > 0$. (In Figure 10.5 symbol E indicates E_1 and symbol p indicates the exchange ratio p_1/p_2).

We immediately see that if the aggregate excess demand function $E(p)$ is continuous and satisfies desirability condition, then there is at least one stable equilibrium point. Moreover, if the economy is regular, then stable and unstable equilibria alternate, the first ones are in number equal to the half of the (uneven) number of equilibria plus ½ and the second ones in number smaller by one than the number of stable equilibria.

Finally, equilibrium that is realized here depends on the first exchange ratio announced by the auctioneer. If there is only one equilibrium exchange ratio, it will always be reached independent of the first announcement of the auctioneer. If there are many equilibria, it leads to the stable equilibrium exchange ratio that is included in the interval between the two unstable equilibrium exchange ratios where the first exchange ratio announcement of the auctioneer belongs (if this exchange ratio is not between two unstable equilibria, the nearest equilibrium exchange ratio is reached). If the auctioneer's first announcement is the exchange ratio that corresponds to an unstable equilibrium, then this unstable equilibrium is realized. This is the only way that this unstable equilibrium can arise.

Conclusions obtained for an economy with only two goods do not hold for economies with more goods. The analysis of those economies is more complex. (The first attempts by Walras, 1900, and Hicks, 1939, were abandoned. They considered a sequence of auctioneer's announcements in discrete time and introduced questionable or inconclusive assumptions. Modern analysis, which is dynamic and in continuous time, derives from the contribution by Samuelson, 1941).

In the stability analysis in continuous time, market behavior (represented by the auctioneer) is usually indicated by a system of differential equations of the type

$$\frac{\mathrm{d}p_h(t)}{\mathrm{d}t} = f_h(E_h(p(t))) \ , \qquad h = 1,\ldots, k$$

where $f_h(.)$ is, for $h = 1,\ldots, k$, a function that keeps the sign of its argument (that is $f_h(a) \geq 0$ if $a \geq 0$ and $f_h(a) \leq 0$ if $a \leq 0$). Equilibrium (x^*, y^*, p^*) is stable if $\lim_{t\to\infty} p(t) = p^*$ for the function $p(t)$ that solves this system. (Note, that this limit, if it exists, necessarily gets the equilibrium vector of prices. In fact, $\lim_{t\to\infty} p(t) = p^*$ implies $\lim_{t\to\infty} \frac{\mathrm{d}p_h(t)}{\mathrm{d}t} = 0$ and so $\lim_{t\to\infty} f_h(E_h(p(t)) = 0$, that is $\lim_{t\to\infty} E_h(p(t)) = 0$, for every $h = 1,\ldots, k$). Stability conditions are represented by the conditions on the functions $f_h(.)$ and $E_h(.)$, for $h = 1,\ldots, k$, that determine the occurrence of stability.

A theory that would explain the preceding behavior of the auctioneer does not exist. In particular, it is not a result of choice. Moreover, there is not any reason why it corresponds to a system of differential equation of first and not higher order. We can only say that this is the simplest system that guarantees the market law. i.e. that the price of a good increases (falls) if its excess demand is positive (negative). This justification explains also why the variation in price of one good depends only on the excess demand for this good and not also on the excess demand for other goods. However, if the aim of the auctioneer was to achieve an equilibrium, a different behavior would be more efficient. The auctioneer could even determine all the equilibria with the information that he can receive through the agents answers to the announced prices.[5]

The kind of behavior which is assumed by the standard analysis of stability corresponds to an auctioneer that keeps track of only the excess demands corresponding to the prices announced last, neglecting all the information obtained before. In other words, we assume that, although the market is organized, it behaves in an elementary way, with bounded rationality. Stability or instability of equilibrium is, obviously, relative to the market behavior that was assumed, expressed by a system of differential equations. With respect to the analogy from the beginning of this paragraph, the difference between equilibrium stability analysis of a material point and stability of competitive equilibrium lays in the availability of a satisfying theory of the material point movements when it is not in the equilibrium position (it moves downwards and dissipating energy makes it converge to the lowest point). On the contrary, a satisfying theory for the movement of economic values when they are not in a competitive equilibrium does not exists. The assumption of the Walrasian auctioneer is only one of the reasonable assumptions and it can be substituted with other rational assumptions that would determine different stability conditions. In other words, we lack a (analytically as well as synthetically) satisfactory theory of market behavior out of equilibrium.

Moreover, with the described behavior, changes in the prices do not maintain, in general, a normalization (that is, we have $p(t)\in\mathbb{R}^k_+$ and not, for example, $p(t)\in S^{k-1}$). A case in which normalization is maintained (with $p(t)\in\{p\in\mathbb{R}^k_+ : \sum_{h=1}^{k} p_h{}^2 = 1\}$) is shown in the next proposition, that is helpful to understand how problematic the existence of stable equilibrium is for $k > 2$ (even when only one equilibrium exists).

[5] If, for example, prices announced by the auctioneer would come from the system of differentiable equations $\frac{\mathrm{d}\tilde{p}(t)}{\mathrm{d}t} = -\lambda\,(\mathrm{D}_{\tilde{p}}\tilde{E}(p(t)))^{-1}\,\tilde{E}(p(t))$, where $\lambda > 0$, $\tilde{E} = [I \vdots 0]E$ and $\tilde{p} = [I \vdots 0]p$, then every equilibrium would be (locally) stable (Smale, 1976). In fact, this process coincides with the one that in mathematics is called Newton method, that aims at determining the solution of a system of equations. Note, on one hand, that price adjustment process depends on the excess demands of the other goods (and not only on the excess demand of the same good) and, on the other hand, that the auctioneer must know, with respect to announced prices, not only excess demand $\tilde{E}(p)$ but also the derivative of this function $\mathrm{D}_{\tilde{p}}\tilde{E}(p)$.

Proposition 12.9 If market behavior is governed by the system $\frac{\mathrm{d}p_h(t)}{\mathrm{d}t} = E_h(p(t))$ for $h = 1,\ldots, k$ (that is, function $f_h(.)$ is the identity function for every $h = 1,\ldots, k$) and $p(0) \in \{p \in \mathbb{R}_+^k : \sum_{h=1}^k p_h^{\ 2} = 1\}$, then $p(t) \in \{p \in \mathbb{R}_+^k : \sum_{h=1}^k p_h^{\ 2} = 1\}$ for every $t > 0$, whatever the function $E(p)$ is.

Proof. Consider the derivative $\frac{\mathrm{d}}{\mathrm{d}t}\sum_{h=1}^k p_h(t)^2$ and apply Walras law. We get $\frac{\mathrm{d}}{\mathrm{d}t}\sum_{h=1}^k p_h(t)^2 = 2\sum_{h=1}^k p_h(t)\frac{\mathrm{d}p_h(t)}{\mathrm{d}t} = 2\sum_{h=1}^k p_h(t)E_h(p(t)) = 0$, that is $\sum_{h=1}^k p_h(t)^2$ is constant for every $t \geq 0$, so $p(t) \in \{p \in \mathbb{R}_+^k : \sum_{h=1}^k p_h^{\ 2} = 1\}$ for every $t \geq 0$. □

In Figures 12.9 and 12.10 we represent trajectories of prices, respectively, for $k = 2$ and $k = 3$, assuming that the aggregate excess demand function $E(p)$ is continuous and satisfies desirability condition and the economy is regular. Note that for $k = 2$ there is always at least one stable equilibrium. In fact, the prices move along the circumference arc, towards the interior from the end points with a speed equal to excess demand. Therefore, since excess demand is a continuous function, there is an odd number N of points in which the speed is equal to zero (these are equilibrium points) and a number equal to $\frac{N+1}{2}$ of them towards which the movement of prices converges (these are stable equilibria). In Figure 12.9 there are two stable and one unstable equilibria. For $k = 3$, on the contrary, all equilibria can turn out to be unstable. In Figure 12.10 there is not any stable equilibrium.

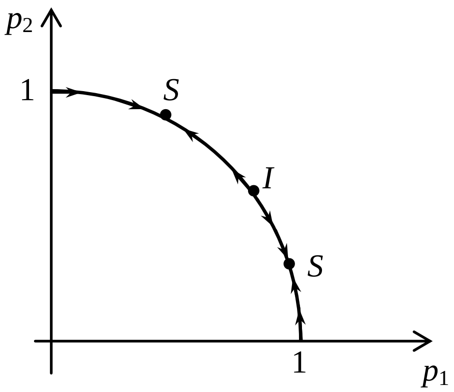

Figure 12.9

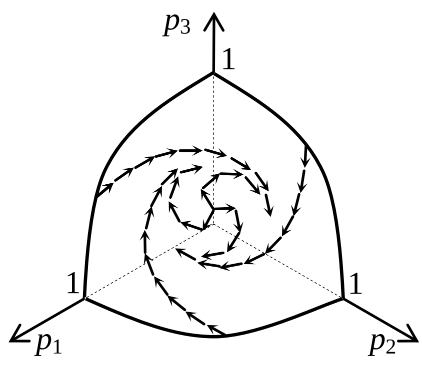

Figure 12.10

Let's consider market behavior as assumed before, specifying the system of differential equations as

$$\frac{\mathrm{d}p_h(t)}{\mathrm{d}t} = \lambda_h E_h(p(t)), \qquad h = 1,\ldots, k-1$$

$$\frac{\mathrm{d}p_k(t)}{\mathrm{d}t} = 0 \ ,$$

where $\lambda_h > 0$ for $h = 1,\dots, k-1$. The last equation indicates that the auctioneer announces always the same price for the k-th good, for example $p_k(t) = 1$, ensuring price normalization. This behavior represents the dynamics of exchange ratios of all goods with respect to the k-th good. As we have already shown, equilibrium (x^*, y^*, p^*) is stable if $\lim_{t\to\infty} p(t) = p^*$ for the function $p(t)$ that solves this system.

We can prove that the equilibrium is globally stable if the aggregate excess demand function $E(p)$ satisfies the weak axiom of revealed preferences or if all the goods are gross substitutes. To prove the first of these propositions we employ a mathematical theorem, called *second method of Lyapunov*.

Second method of Lyapunov. Consider an autonomous system of first order differential equations

$$\frac{\mathrm{d}y_i(t)}{\mathrm{d}t} = f_i(y(t)) \ , \qquad i = 1,\dots, n,$$

with $f_i(y^*) = 0$ for $i = 1,\dots, n$, where $y(t) = (y_1(t),\dots, y_n(t))$. Equilibrium y^* is stable if there exists a function $V:\mathbb{R}^n \to \mathbb{R}$, of the type $V(y-y^*)$ with continuous first order partial derivatives, that satisfies the following three conditions:

a) $V(y-y^*) > 0$ for every $y \neq y^*$ and $V(0) = 0$;

b) $V(y-y^*) \to +\infty$ for $\|y-y^*\| \to +\infty$;

c) $\frac{\mathrm{d}V(y(t)-y^*)}{\mathrm{d}t} < 0$ for every $y \neq y^*$ and $\frac{\mathrm{d}V(y(t)-y^*)}{\mathrm{d}t} = 0$ for $y = y^*$.

Proposition 12.10 If the aggregate demand function $E(p)$ of a regular economy satisfies the desirability condition for all the goods and the weak axiom of revealed preferences holds, then equilibrium is globally stable with respect to the dynamic process represented by the system of equations

$$\frac{\mathrm{d}p_h(t)}{\mathrm{d}t} = \lambda_h E_h(p(t)), \qquad h = 1,\dots, k-1,$$

$$\frac{\mathrm{d}p_k(t)}{\mathrm{d}t} = 0,$$

where $\lambda_h > 0$ for $h = 1,\dots, k-1$.

Proof. The weak axiom of revealed preferences requires, as stated in Proposition 12.8, that inequalities $pE(p') \le 0$ and $E(p') \neq E(p)$ imply $p'E(p) > 0$. Therefore, if $E(p^*) = 0$, then $p^*E(p) > 0$ for every $E(p) \neq 0$. Moreover, by Proposition 12.8, there is a unique vector of equilibrium prices in S^{k-1}, which is positive by desirability condition. Then, there is a unique $p^* \in \mathbb{R}^k_+$, with $p_k^* = 1$, for which $E(p^*) = 0$. Let's introduce the function $V = \sum_{h=1}^{k-1} \frac{1}{\lambda_h}(p_h(t) - p_h^*)^2 + (p_k(t) - p_k^*)^2$. Let's check whether this function satisfies all three conditions required by the second method of Lyapunov, and so it is a *Lyapunov function*. The first two conditions are immediately satisfied. Consider the third condition, apply Walras law and recall that $p_k(t) = p_k^* = 1$. We then get

$$\frac{\mathrm{d}V}{\mathrm{d}t} = \sum_{h=1}^{k-1} \frac{2}{\lambda_h} (p_h(t) - p_h^*) \frac{\mathrm{d}p_h(t)}{\mathrm{d}t} =$$

$$2\sum_{h=1}^{k-1} (p_h(t) - p_h^*) E_h(p(t)) = 2\sum_{h=1}^{k} (p_h(t) - p_h^*) E_h(p(t)) = -2p^* E(p(t))\,.$$

Thus $\frac{\mathrm{d}V}{\mathrm{d}t} < 0$ for every $E(p) \neq 0$ and $\frac{\mathrm{d}V}{\mathrm{d}t} = 0$ if and only if $p = p^*$ (keeping in mind that we consider only price vectors with $p_k(t) = p_k^* = 1$). Then by the second method of Lyapunov, the dynamic process converges towards p^* and as a result it is a globally stable equilibrium. □

Proposition 12.11 A competitive equilibrium of a regular economy, characterized by a continuous and differentiable excess demand function that satisfies desirability condition for all the goods and with only goods that are, in equilibrium, gross substitutes with respect to the other goods (that is. by Definition 12.3, such that $\mathrm{D}_{p_t} E_h(p^*) > 0$ for every $h,t = 1,\ldots, k$ with $h \neq t$ and for every $p^* \in \mathbb{R}_+^k$ with $E(p^*) = 0$) is locally stable with respect to the dynamic process represented by the system of differential equations

$$\frac{\mathrm{d}p_h(t)}{\mathrm{d}t} = \lambda_h E_h(p(t))\,, \qquad\qquad h = 1,\ldots, k-1,$$

$$\frac{\mathrm{d}p_k(t)}{\mathrm{d}t} = 0\,,$$

where $\lambda_h > 0$ for $h = 1,\ldots, k-1$.

Proof. Local stability requires that the prices converge towards its equilibrium values in the neighborhood of p^*. We can, then, linearize the system of the examined differential system around p^*. Keeping in mind that $p_k(t) = p_k^* = 1$, we get

$$\frac{\mathrm{d}\tilde{p}(t)}{\mathrm{d}t} = \hat{\lambda}\ \mathrm{D}_{\tilde{p}}\tilde{E}(p^*)\ (\tilde{p}(t) - \tilde{p}^*)\,,$$

where $\tilde{E} = [I \vdots 0]E$, $\tilde{p} = [I \vdots 0]p$ and $\hat{\lambda}$ is the diagonal matrix with non-zero elements equal to λ_h, for $h = 1,\ldots, k-1$. The solution of this system converges (towards $\tilde{p}^*$) if all the eigenvalues of the matrix $\hat{\lambda}\,\mathrm{D}_{\tilde{p}}\tilde{E}(p^*)$ and, consequently, since $\lambda_h > 0$ for every $h = 1,\ldots, k-1$, of the matrix $\mathrm{D}_{\tilde{p}}\tilde{E}(p^*)$ have a negative real part. A sufficient condition that gives rise to this property is that the matrix $\mathrm{D}_{\tilde{p}}\tilde{E}(p^*)$ has all the elements on the main diagonal negative and positive elsewhere and that there exists a vector $h \in \mathbb{R}_+^{k-1}$ for which $\mathrm{D}_{\tilde{p}}\tilde{E}(p^*)\,h < 0$ (Hahn, 1958 and 1982). The first of these two requirements is satisfied by the assumption that all the goods are gross substitutes. The second one has already been proved with respect to the Proposition 12.7. In fact, we have shown that $(\mathrm{D}_{\tilde{p}}\tilde{E}(p^*))^{\mathrm{T}}\ \tilde{p}^* << 0$. (Recall that transposition does not change the eigenvalues, that is that the matrices $\mathrm{D}_{\tilde{p}}\tilde{E}(p^*)$ and $(\mathrm{D}_{\tilde{p}}\tilde{E}(p^*))^{\mathrm{T}}$ have the same eigenvalues). Therefore, the unique existing equilibrium (by Proposition 12.7) is locally stable (but, for $k > 2$, not necessarily globally). Note that this result holds for any positive parameters λ_h, with $h = 1,\dots, k-1$. □

12.4 Equilibrium stability in real time (without *tâtonnements*)

Equilibrium stability analysis carried out in the preceding paragraph is based on the condition that no exchanges or production processes take place in the contracting phase. It leads to propositions (like Propositions 12.10 and 12.11) that ensure stability under very strong conditions. These assumptions are a bit weaker if we allow for exchanges and production to take place out of equilibrium. Naturally, these actions modify the economy (at least by changing agents' endowments), so eventual convergence is possible towards a competitive equilibrium of an economy generally different from the initial one.

In what follows, we will present two possible dynamic processes. In the first one (*Edgeworth process*) the exchanges are made under barter (without price formalization); in the second one (*Hahn process*) the prices are equal for all the agents, so out of equilibrium the exchanges are, in general, rationed.

For simplicity, we consider only pure exchange economies (so without production) with only durable goods. (In order to get the idea, imagine for example an economy of children that exchange football players figures or an economy of financiers that exchange securities). Moreover, the exchanges modify only the endowments of the agents but not their preferences or their number.

We start from an economy $\mathcal{E} = (\mathbb{R}^k_+, \succsim_i\rangle, \omega_i,\ i = 1,\dots, n)$. Since the goods are durable, the exchanges modify the endowments of the agents but not the total quantity of the goods. That is, the economy changes with time and can be described as $\mathcal{E}(t) = (\langle \mathbb{R}^k_+, \succsim_i\rangle, \omega_i(t),\ i = 1,\dots, n)$, with $\Sigma_{i=1}^n \omega_i(t) = \Sigma_{i=1}^n \omega_i(0) = \Omega$ for every $t > 0$. (In what follows, we assume, as in general in Paragraph 12.3, that these economies are regular, all the goods are desirable and $\omega_i > 0$ for $i = 1,\dots, n$, so every agent has a positive endowment of at least one good). There is stability if the exchange process leads, when t increases, to an allocation $(\omega_i^*)_{i=1}^n$ that is a competitive equilibrium for the economy $\mathcal{E}^* = (\langle \mathbb{R}^k_+, \succsim_i\rangle, \omega_i^*, i = 1,\dots, n)$.

Edgeworth process. In this process the agents exchange goods if the exchange is favorable for each of them. We assume that information system is such that it signals the existence of favorable exchanges, so the exchange process continues until there are no more possible favorable exchanges.

Naturally, there is a multiplicity of possible sequences of exchanges and each of them leads, in general, to a different allocation if it converges. We will now examine whether all these sequences converge and if each of them converges to a competitive equilibrium allocation.

Proposition 12.12 If agents' systems of preference are regular, continuous, strongly monotonic and strictly convex and the sequence of exchanges follows Edgeworth process, then the allocation $(\omega_i(t))_{i=1}^n$ of the economy $\mathcal{E}(t) = (\langle \mathbb{R}_+^k, \succsim_i \rangle, \omega_i(t), i = 1,\ldots, n)$ converges towards a competitive equilibrium allocation $(\omega_i^*)_{i=1}^n$.

Proof. The preferences of each agent are regular and continuous so they can be represented with a continuous utility function (Proposition 3.2). Moreover, this function is strongly increasing (since the preferences are strongly monotonic), bounded from above (since the available quantity of the goods is limited) and strictly quasi-concave (since the preferences are strictly convex). Moreover, the utility possibility set (Definition 8.6)

$$U = \{u \in \mathbb{R}^n : u_i = u_i(\omega_i) \text{ for } i = 1,\ldots, n \text{ with } \Sigma_{i=1}^n \omega_i = \Omega\}$$

is non-empty, closed and bounded from above (Proposition 8.4). Every Edgeworth process determines a sequence of utility vectors such that $u(t') \geq u(t)$ for every pair t, t' with $t' > t$, moreover with $u(t') > u(t)$ if favorable exchanges are possible from $(\omega_i(t))_{i=1}^n$ and with $u(t') = u(t)$ if $(\omega_i(t))_{i=1}^n$ is efficient, i.e. $u(t) \in U_{max} = \{u \in U$: there does not exist $u' \in U$ such that $u' > u\}$. If the process is continuous, that is allocation $(\omega_i(t))_{i=1}^n$ is a continuous function of t, then function $\Sigma_{i=1}^n u_i(t)$ is continuous, bounded from above and monotonically non decreasing: in particular, it is increasing if $u(t) \in U \setminus U_{max}$ and constant if $u(t) \in U_{max}$. As a result, there exists a bounded $\lim_{t\to\infty} \Sigma_{i=1}^n u_i(t)$. Therefore, since $u(t') \geq u(t)$ for every pair t, t' with $t' > t$, there exists a bounded $\lim_{t\to\infty} u(t)$ of the utility vector, necessarily with $\lim_{t\to\infty} u(t) \in U_{max}$. Recalling that we have by Definition 8.5

$$U_{max} = \{u \in U: u_i = u_i(\omega_i) \text{ for } i = 1,\ldots, n \text{ with } \Sigma_{i=1}^n \omega_i = \Omega \text{ and } (\omega_i)_{i=1}^n \in PO\}$$

we get that $\lim_{t\to\infty} (\omega_i(t))_{i=1}^n$ is an efficient allocation (there is a unique allocation since strict convexity of preferences prevents us from having two allocations equally preferred by all the agents). Finally, since Edgeworth process determines the convergence towards an efficient allocation, we get that this allocation is a competitive equilibrium allocation by the second welfare theorem . □

In Figures 12.11 and 12.12, for an economy with two agents and two goods, we show a possible path that follows Edgeworth process, respectively, in Edgeworth-Pareto box diagram and in utility space.

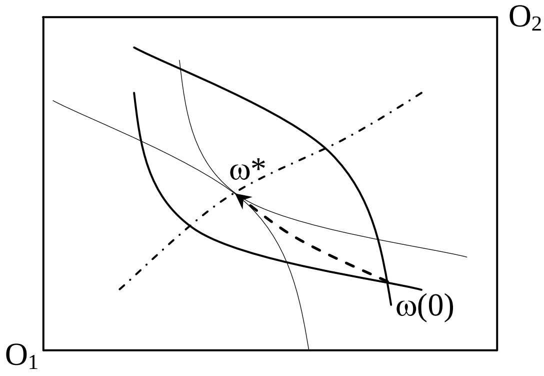

Figure 12.11

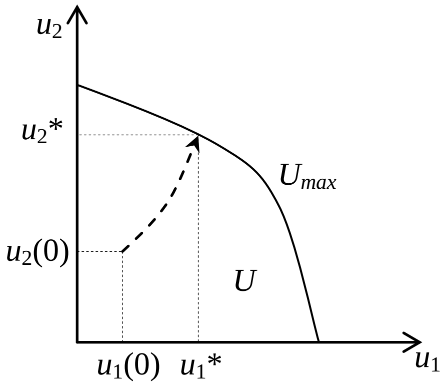

Figure 12.12

Hahn process. This process is different from the process directed by the Walrasian auctioneer described in Paragraph 12.3 only because there are exchanges even if the prices announced by auctioneer do not form an equilibrium. Auctioneer raises prices of the goods that have positive aggregate excess demand and reduces the prices of goods with negative aggregate excess demand. The presence of auctioneer assures that prices are equal for all the agents. The agents form their demand and supply with respect to those announcements.[6] Out of equilibrium, aggregate excess demands are not equal to zero, so not all the asked exchanges can be executed. Therefore, it is necessary to specify which of the exchanges are realized, in other words, we need to come up with an exchange rationing scheme. In what follows we adopt the following assumption: for every good for which aggregate excess demand is positive (that is, demand exceeds supply), agents with non positive excess demand sell the desired quantity of good, while those with positive excess demand are rationed. They buy less than desired of the good, such that the sum of the total quantity bought is equal to the total supply from the sellers. By analogy, for every good with negative excess demand. We will now examine whether the sequence of exchanges determined by such rationing converges and if it converges to a competitive equilibrium allocation.

Proposition 12.13 If the systems of preferences of the agents are regular, continuous, strongly monotone and strictly convex, the auctioneer announces positive prices for all goods (keeping in mind that the goods are desirable) and the sequence of exchanges follows Hahn process, then the

[6] In what follows, we assume that agents form their demand and supply without realizing that they may be rationed. This simplifies analysis, but is in conflict with rationality of agents. In fact, if an agent realizes that a certain good will be rationed (so he will be able to, for example, buy only half of what he asks) he may want to demand a quantity higher than desired (for example doubled) in order to get a quantity close to the desired one (on this point and other aspects, see Fisher, 1983). For the equilibrium stability in general, see Hahn (1982).

allocation $(\omega_i(t))_{i=1}^{n}$ of the economy $\mathcal{E}(t) = (\langle \mathbb{R}_+^k, \succsim_i\rangle, \omega_i(t), i = 1,\ldots, n)$ converges towards a competitive equilibrium allocation $(\omega_i{*})_{i=1}^{n}$.

Proof. With the indicated assumptions, every agent in economy $\mathcal{E}(t)$ has a demand function $d_{ih}(t) = d_{ih}(p(t), p(t)\omega_i(t))$ that is continuous (if $p(t)\omega_i(t) > 0$) for every good $h = 1,\ldots, k$, with $d_i(t) > 0$. Aggregate demand function for the h-th good is $E_h(t) = \Sigma_{i=1}^{n} d_{ih}(p(t), p(t)\omega_i(t)) - \Omega_h$. If $E_h(t) = 0$, then none of the agents is rationed with respect to the h-th good, so we have $\omega_{ih}(t) = d_{ih}(t)$ for every $i\in N$. The auctioneer does not change the price of this good, that is $\frac{\mathrm{d}p_h(t)}{\mathrm{d}t} = 0$. If $E_h(t) > 0$, the set of agents N is divided into two subsets: subset $I_h^{nr}(t)$ of the non rationed agents (those with non positive excess demand) and the subset $I_h^{r}(t)$ of rationed agents (those with positive excess demand). We then get $\omega_{ih}(t) = d_{ih}(t)$ for every $i\in I_h^{nr}(t)$ and $\omega_{ih}(t) < d_{ih}(t)$ for every $i\in I_h^{r}(t)$. The auctioneer raises the price of this good, that is $\frac{\mathrm{d}p_h(t)}{\mathrm{d}t} > 0$. Analogously, if $E_h(t) < 0$, the set of agents N is divided into two subsets: subset $I_h^{nr}(t)$ of non rationed agents (those with non negative excess demand) and subset $I_h^{r}(t)$ of rationed agents (those with negative excess demand). We then have $\omega_{ih}(t) = d_{ih}(t)$ for every $i\in I_h^{nr}(t)$ and $\omega_{ih}(t) > d_{ih}(t)$ for every $i\in I_h^{r}(t)$. Auctioneer lowers the price of this good, that is $\frac{\mathrm{d}p_h(t)}{\mathrm{d}t} < 0$. Note that in each case, with or without rationing, we must have $\Sigma_{h=1}^{k} p_h(t)\frac{\mathrm{d}\omega_{ih}(t)}{\mathrm{d}t} = 0$ for every $i\in N$, because the exchange leaves the value of the endowment unchanged. Note, moreover, that all the agents (rationed or not) are never harmed by the exchange, that is $u_i(\omega_i(t))$ is a non decreasing function of t. Therefore, since $\omega_i(0) > 0$, and also $\omega_i(t) > 0$ for every $t > 0$ we get $p(t)\omega_i(t) > 0$ (in this way we are certain that every agent always has a continuous excess demand function). However, while the direct utility function is a non decreasing function of t, the opposite happens for the indirect utility function $u_i{*}(p(t), p(t)\omega_i(t))$, which is, as we will see soon, a non increasing function of t. In fact, we get

$$\frac{\mathrm{d}u_i{*}(t)}{\mathrm{d}t} = \sum_{h=1}^{k} \frac{\partial u_i{*}}{\partial p_h(t)} \frac{\mathrm{d}p_h(t)}{\mathrm{d}t} + \frac{\partial u_i{*}}{\partial(p(t)\omega_i(t))} \sum_{h=1}^{k} (\omega_{ih}(t)\frac{\mathrm{d}p_h(t)}{\mathrm{d}t} + p_h(t)\frac{\mathrm{d}\omega_{ih}(t)}{\mathrm{d}t})$$

Therefore, keeping in mind, on one hand, that $\Sigma_{h=1}^{k} p_h(t)\frac{\mathrm{d}\omega_{ih}(t)}{\mathrm{d}t} = 0$ and, on the other hand, of Antonelli-Roy relationship (Proposition 3.13) according to which

$$\frac{\partial u_i{*}}{\partial p_h(t)} = -d_{ih}(t)\frac{\partial u_i{*}}{\partial(p(t)\omega_i(t))},$$

we get

$$\frac{\mathrm{d}u_i*(t)}{\mathrm{d}t} = -\frac{\partial u_i*}{\partial(p(t)\omega_i(t))}\sum_{h=1}^{k}(d_{ih}(t)-\omega_{ih}(t))\frac{\mathrm{d}p_h(t)}{\mathrm{d}t}.$$

Since $\frac{\partial u_i*}{\partial(p(t)\omega_i(t))} > 0$ and $(d_{ih}(t)-\omega_{ih}(t))\frac{\mathrm{d}p_h(t)}{\mathrm{d}t} \geq 0$ (in particular, if an agent is not rationed for the h-th good then $\omega_{ih}(t) = d_{ih}(t)$, if he is rationed and $E_h(t) > 0$ then $\omega_{ih}(t) < d_{ih}(t)$ and $\frac{\mathrm{d}p_h(t)}{\mathrm{d}t} > 0$, and if he is rationed and $E_h(t) < 0$ then $\omega_{ih}(t) > d_{ih}(t)$ and $\frac{\mathrm{d}p_h(t)}{\mathrm{d}t} < 0$), we get $\frac{\mathrm{d}u_i*(t)}{\mathrm{d}t} \leq 0$, with $\frac{\mathrm{d}u_i*(t)}{\mathrm{d}t} < 0$ if the i-th agent is rationed for some good and $\frac{\mathrm{d}u_i*(t)}{\mathrm{d}t} = 0$ if he is not rationed for any good. As a consequence, function $\Sigma_{i=1}^{n} u_i*(t)$ is continuous and monotonically non increasing: in particular it is decreasing if some good is not in equilibrium and it is constant if all the goods are in equilibrium. Moreover, this function is bounded from below (to account for this, it is enough to choose utility functions such that $u_i(0) = 0$ for every $i = 1,\ldots, n$, and note that it must be that $u_i*(t) \geq 0$ for every t). Consequently, $\lim_{t\to\infty}\Sigma_{i=1}^{n}\frac{\mathrm{d}u_i*(t)}{\mathrm{d}t} = 0$. Therefore, for every $i = 1,\ldots, n$, since $\frac{\mathrm{d}u_i*(t)}{\mathrm{d}t} \leq 0$ we get $\lim_{t\to\infty}\frac{\mathrm{d}u_i*(t)}{\mathrm{d}t} = 0$, so the economy converges towards a situation without rationing, that is towards a competitive equilibrium. □

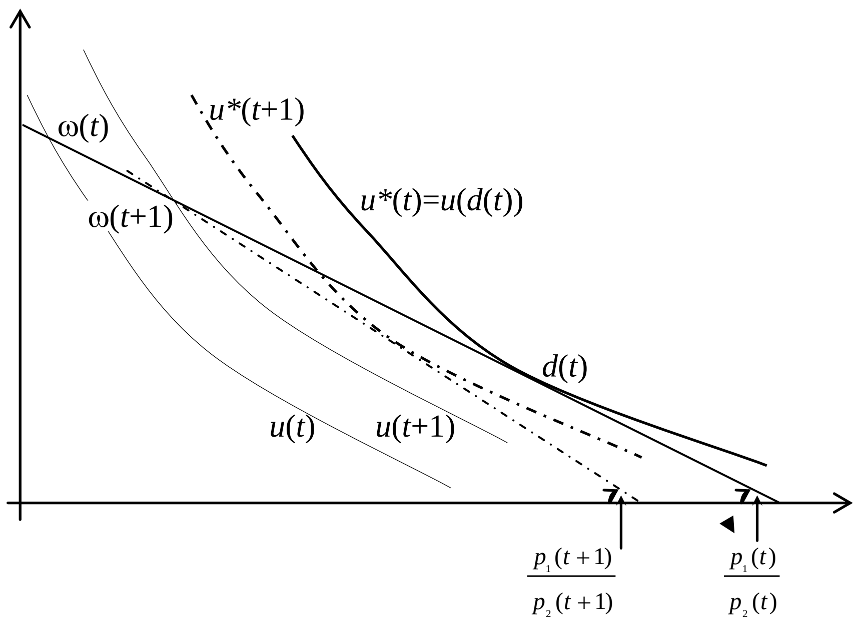

Figure 12.13

In Figure 12.13 we illustrate how rationing reduces indirect utility (while direct utility increases) in an economy with two goods.

12.5 Comparative statics of general competitive equilibrium

Comparative statics analysis of competitive equilibrium (introduced in Paragraph 10.4 for partial equilibrium) studies the relationship between equilibria of different economies, that is the dependence of an equilibrium on the data of the economy. In the examined case, how a change in preferences, technology or quantity of available resources modifies the equilibrium represented by prices and allocation of goods.

Let the competitive equilibrium be determined by the following conditions: $x_i \in d_i(p;\ m_i)$, $y_j \in s_j(p)$ and $\sum_{i=1}^{n} x_i \le \sum_{i=1}^{n} \omega_i + \sum_{j=1}^{m} y_j$, where $m_i = p\omega_i + \sum_{j=1}^{m} \theta_{ij} \pi_j{}^*$, with $\pi_j{}^* = \max_{y_j \in Y_j} p y_j$, $d_i(p;\ m_i) = \arg \max_{x_i \in \{x_i \in X_i : p x_i \le m_i\}} u_i(x_i)$ and $s_j(p) = \arg \max_{y_j \in Y_j} p y_j$, for $i = 1,\dots, n$ and $j = 1,\dots, m$. Equilibrium allocation and prices (x^*, y^*, p^*) (where $x^* = (x_i^*)_{i=1}^{n}$ and $y^* = (y_j^*)_{j=1}^{m}$) turn out to depend on functions $u_i(.)$, endowments (ω_i, θ_{ij}) and sets Y_j, with $i = 1,\dots, n$ and $j = 1,\dots, m$. If we represent these data by a vector of parameters $\alpha = (\alpha_1,\dots, \alpha_t)$, then the competitive equilibrium (x^*, y^*, p^*) is a function of α. We analyze this function to get comparative statics for general competitive equilibrium.

The comparison can either look at discrete changes in parameters $\alpha_b - \alpha_a$ or continuous changes $d\alpha$. If there are many equilibria, then global comparative statics analysis (that is, the one that examines the effects of discrete changes in parameters) is not very significant: we get the change of the set of equilibria. On the contrary, local analysis (that is the one that examines the effects of infinitesimal changes in parameters) is significant when equilibria are isolated (i.e., if the economy is regular), with notice that static analysis considers an infinitesimal change of the equilibrium. The derivative $\frac{\partial \beta}{\partial \alpha}$, where β is a variable and α is a given, represents the change in the neighborhood of a single generic equilibrium.

Suppose that we are interested in studying comparative statics of equilibrium prices highlighting their dependence on a vector α of data. Since prices are determined by condition $E(p;\ \alpha) = 0$ (because the aggregate excess demand function is single-valued and desirability conditions are satisfied), or better, as shown in Paragraph 12.1 (right after Definition 12.1) by the system of k–1 equations $\tilde{E}(p;\alpha) = 0$, we get, differentiating these equations,

$$\mathrm{D}_\alpha \tilde{p}^*(\alpha) = -(\mathrm{D}_{\tilde{p}} \tilde{E}(p;\alpha))^{-1} \mathrm{D}_\alpha \tilde{E}(p;\alpha) .$$

In what follows, we examine the dependence of prices on quantity of resources. This analysis intends whether the competitive equilibrium prices are *indices of relative scarcity.* In other words, whether an increase of available quantity of one good, when the quantity of the other goods remains unchanged, reduces its relative price (that is, the ratios between the price of the good under examination and the prices of the other goods).[7]

If there are no free goods and excess demand functions are single-valued, then the competitive equilibrium of economy $\mathcal{E} = (\langle X_i, \succsim_i \rangle, Y_j, \omega_i, \theta_{ij}, i = 1,\ldots,n,\ j = 1,\ldots,m)$ with *free disposal* is determined by condition $E(p^*,\omega)=0$ (where dependence on the quantity of resources $\omega = (\omega_{11},\ldots,\omega_{1k},\ldots,\omega_{n1},\ldots,\omega_{nk}) \in \mathbb{R}^{nk}_{++}$ is highlighted), with $E(p,\omega) = \sum_{i=1}^{n}(d_i(p,m_i)-\omega_i)-\sum_{j=1}^{m}s_j(p)$, where $m_i = p\omega_i + \sum_{j=1}^{m}\theta_{ij}ps_j(p)$. If aggregate excess demand function $E(p;\omega)$ is differentiable with respect to both prices and quantity of resources, local comparative statics analysis is represented by the derivatives of equilibrium exchange ratios with respect to quantity of resources. That is, its purpose is to determine the derivatives $\dfrac{\partial \frac{p_h^*}{p_s^*}}{\partial \omega_{ir}}$, for $i = 1,\ldots, n$ and $h,s,r = 1,\ldots, k$ with $h \neq s$, from the equilibrium condition $E(p^*; \omega) = 0$.

We now need to extend the definitions of gross substitutes, introduced in Chapter 3 (Definition 3.5 and 10.3), and of normal goods (Definition 3.6).

Definition 12.4 (*Normal goods and gross substitutes for aggregate excess demand function*) The h-th good is normal if $\dfrac{\partial E_h(p;\omega)}{\partial \omega_{ir}} > 0$ for every pair i,r, with $r \neq h$, $r = 1,\ldots, k$ and $i = 1,\ldots, n$; the h-th good is gross substitute with respect to the t-th good, with $h,t = 1,\ldots, k$ and $h \neq t$, if $\dfrac{\partial E_h(p;\omega)}{\partial p_t} > 0$.

With respect to normal goods, Walras law requires $pE(p;\omega) = 0$, so $\sum_{h=1}^{k} p_h \dfrac{\partial E_h(p;\omega)}{\partial \omega_{ir}} = 0$ for $i = 1,\ldots, n$ and $r = 1,\ldots, k$. Then, if $\dfrac{\partial E_h(p;\omega)}{\partial \omega_{ir}} > 0$ for every $r \neq h$, we get $\dfrac{\partial E_h(p;\omega)}{\partial \omega_{ih}} < 0$ and $\sum_{h=1,h\neq s}^{k} p_h \dfrac{\partial E_h(p;\omega)}{\partial \omega_{ir}} < 0$ for $r \neq s$, $i = 1,\ldots, n$ and $h = 1,\ldots, k$, since $\sum_{h=1,h\neq s}^{k} p_h \dfrac{\partial E_h(p;\omega)}{\partial \omega_{ir}} + p_s \dfrac{\partial E_s(p;\omega)}{\partial \omega_{ir}} = 0$ and $\dfrac{\partial E_s(p;\omega)}{\partial \omega_{ir}} > 0$. Moreover, since the excess demand functions are

[7] More extensive comparative statics analysis can be found in Montesano (1993). The relation between scarcity and prices is discussed in Montesano (1995).

homogenous of degree zero with respect to prices (that is, $E(\alpha p;\omega) = E(p;\omega)$ for every $\alpha > 0$), we get for every $\alpha > 0$ that $\frac{\partial E_h(\alpha p;\omega)}{\partial \omega_{ir}} = \frac{\partial E_h(p;\omega)}{\partial \omega_{ir}} > 0$ for $h \neq r$ if the good is normal. Then, since $E(p;\omega) = \sum_{i=1}^{n}(d_i(p,m_i) - \omega_i) - \sum_{j=1}^{m} s_j(p)$, where $m_i = p\omega_i + \sum_{j=1}^{m}\theta_{ij} p s_j(p)$ and there are no free goods (so $p_r > 0$), the h-th good, if it is normal for all the consumers according to Definition 3.6, that is if $\frac{\partial d_{ih}(p,m_i)}{\partial m_i} > 0$ for every $i = 1,\ldots, n$, then it is also normal according to the definition 12.4. In fact, we get $\frac{\partial E_h(p;\omega)}{\partial \omega_{ir}} = \frac{\partial d_{ih}(p,m_i)}{\partial m_i} p_r > 0$ for $h \neq r$.

With respect to gross substitutability, Walras law requires $pE(p;\omega) = 0$, so $\sum_{h=1}^{k} p_h \frac{\partial E_h(p;\omega)}{\partial p_t} + E_t(p;(\omega_i)_{i=1}^{n}) = 0$. Then, in equilibrium we get $\sum_{h=1}^{k} p_h{*} \mathrm{D}_{p_t} E_h(p{*};\omega) = 0$ because $E_t(p{*};\omega) = 0$. Then, the h-th good, if it is gross substitute with respect to the other goods, that is $\mathrm{D}_{p_t} E_h(p{*};\omega) > 0$ for every $t \neq h$, is also an ordinary good, that is $\mathrm{D}_{p_h} E_h(p{*};\omega) < 0$. By the homogeneity of degree zero of the excess demand functions we get $\mathrm{D}_{\alpha p_t} E_h(\alpha p{*};\omega) = \alpha^{-1}\, \mathrm{D}_{p_t} E_h(p{*};\omega)$ for every $\alpha > 0$, so $\mathrm{D}_{p_t} E_h(p{*};\omega) > 0$ if and only if $\mathrm{D}_{\alpha p_t} E_h(\alpha p{*};\omega) > 0$ and $\mathrm{D}_{\alpha p_t} E_h(\alpha p{*};\omega) > 0$ for every $t \neq h$ implies $\mathrm{D}_{\alpha p_h} E_h(\alpha p{*};\omega) < 0$. Finally, leaving aside the derivatives $\mathrm{D}_p s_j(p)$ (absent if we consider a pure exchange economy), the h-th good, if it is normal for all the consumers according to Definition 3.6 and is a gross substitute for all the consumers with respect to the t-th good according to Definition 3.5, is also a gross substitute with respect to the t-th good according to Definition 12.4. In fact, we get $\frac{\partial E_h(p;\omega)}{\partial p_t} = \sum_{i=1}^{n}\left(\frac{\partial d_{ih}(p,m_i)}{\partial p_t} + \frac{\partial d_{ih}(p,m_i)}{\partial m_i}\omega_{it}\right) > 0$ for every $t = 1,\ldots, k$ with $t \neq h$.

Now, we are ready to introduce a following proposition.

Proposition 12.14 Competitive equilibrium prices are indices of relative scarcity if all the goods are gross substitutes and normal (according to Definition 12.4), that is, the exchange ratios determined by the equilibrium condition $E(p{*};(\omega_i)_{i=1}^{n}) = 0$ have derivatives $\frac{\partial\, p_h{*}/p_s{*}}{\partial \omega_{ih}} < 0$, for every $i = 1,\ldots, n$ and $h,s = 1,\ldots, k$ with $h \neq s$.

Proof. Condition $E(p{*};(\omega_i)_{i=1}^{n}) = 0$ determines equilibrium exchange ratios only if the Jacobian matrix of function $E(p;(\omega_i)_{i=1}^{n})$ with respect to the prices, evaluated at equilibrium prices, that is matrix $D_pE(p^*;(\omega_i)_{i=1}^n)$, is of rank k–1. It cannot be of rank equal to k because Walras law, according to which $pE(p;(\omega_i)_{i=1}^n) = 0$, implies that aggregate excess functions are linearly dependent. Proposition 12.5 indicates that the property according to which the rank is equal to k–1 holds generically, that is for almost every $(\omega_i)_{i=1}^n >> 0$. For every given s-th good we consider equilibrium conditions of the other goods $E_h(p^*;(\omega_i)_{i=1}^n) = 0$, where $h = 1,\ldots, k$ with $h \neq s$, and denote with $\tilde{E}$ and $\tilde{p}$ the vectors, with k–1 components, (E_h) and $(\frac{p_h}{p_s*})$, where $h = 1,\ldots, k$ with $h \neq s$. Now, recalling that excess demand functions are homogenous of degree zero with respect to prices, we get the exchange ratios $\tilde{p}*$ as a function of resources $(\omega_i)_{i=1}^n$ from equilibrium condition

$$\tilde{E}(\tilde{p}*;(\omega_i)_{i=1}^n) = 0 .$$

Deriving this condition with respect to ω_{ir} , we get the relationship

$$D_{\tilde{p}}\tilde{E}\ D_{\omega_{ir}}\tilde{p}* + D_{\omega_{ir}}\tilde{E} = 0$$

(where, for simplicity of notation, symbols $D_{\tilde{p}}\tilde{E}$ and $D_{\omega_{ir}}\tilde{E}$ respectively denote $D_{\tilde{p}}\tilde{E}(\tilde{p}^*;(\omega_i)_{i=1}^n)$ and $D_{\omega_{ir}}\tilde{E}(p^*;(\omega_i)_{i=1}^n)$), from which we get

$$D_{\omega_{ir}}\tilde{p}^* = -(D_{\tilde{p}}\tilde{E})^{-1}\ D_{\omega_{ir}}\tilde{E}$$

where $D_{\tilde{p}}\tilde{E}$ is a full rank $(k–1)\times(k–1)$ Jacobian matrix and $(D_{\tilde{p}}\tilde{E})^{-1}$ is its inverse and $D_{\omega_{ir}}\tilde{E}$ is the vector of the derivatives of $\tilde{E}(\tilde{p}^*;(\omega_i)_{i=1}^n)$ with respect to ω_{ir}. This relationship can be written as

$$D_{\omega_{ir}}\tilde{p}^* = -\hat{P}^{-1}\ \hat{B}^{-1}\ G\ z_{ir}$$

where $\hat{P}$ is a diagonal matrix of the same elements $\tilde{p}*$; $\hat{B}$ is a diagonal matrix composed of the main diagonal of the matrix $D_{\tilde{p}}\tilde{E}$, that is with elements $D_{\tilde{p}_h}\tilde{E}_h = p_s*\ D_{p_h}E_h$, where $h = 1,\ldots, k$ and $h \neq s$; $G = \hat{B}\hat{P}(D_{\tilde{p}}\tilde{E})^{-1}\hat{P}^{-1}$; and $z_{ir} = \hat{P}\ D_{\omega_{ir}}\tilde{E}$. The elements on the main diagonal of matrix $\hat{P}$ are all positive because there are no free goods and those of matrix $\hat{B}$ are all negative because the goods are gross substitutes. The vector z_{ir} is composed of $z_{h,ir} = \frac{p_h*}{p_s*}\frac{\partial\tilde{E}_h}{\partial\omega_{ir}} = \frac{p_h*}{p_s*}\frac{\partial E_h}{\partial\omega_{ir}}$ with $h = 1,\ldots, k$ and $h \neq s$, that are positive for $h \neq r$ and negative for $h = r$, because the goods are normal (for simplicity of notation symbol $\frac{\partial E_h}{\partial\omega_{ir}}$ denotes $\frac{\partial E_h(p^*;(\omega_i)_{i=1}^n)}{\partial\omega_{ir}}$ and $D_{p_t}E_h$ denotes $D_{p_t}E_h(p^*;(\omega_i)_{i=1}^n)$). Matrix $F = G^{-1} = \hat{P}\ D_{\tilde{p}}\tilde{E}\ \hat{P}^{-1}\ \hat{B}^{-1}$ is composed of $f_{ht} = \frac{p_h*}{p_t*}\frac{D_{p_t}E_h}{D_{p_t}E_t}$, so assumption that goods are gross substitutes requires $f_{tt} = 1$ and $f_{ht} < 0$ for $h \neq t$, with $\sum_{h=1}^{k} f_{ht} = 0$, $\sum_{h=1,h\neq t}^{k} f_{ht} = -1$ and $\sum_{h=1,h\neq t,s}^{k} f_{ht} > -1$. Then, matrix $A = I - F$ is semipositive, with elements a_{ht} such that $\sum_{h=1,h\neq t,s}^{k} a_{ht} = -\sum_{h=1,h\neq t,s}^{k} f_{ht} < 1$. Therefore, we apply Metzler theorem (1951) according to which matrix $G = F^{-1} = (I - A)^{-1}$ is positive with elements $g_{hh} > g_{ht} > 0$ for $h \neq t$ and $h,t = 1,\ldots, k$, with $h,t \neq s$. Therefore, since $\mathrm{D}_{\omega_{ir}} \tilde{p}^* = -\hat{P}^{-1}\, \hat{B}^{-1}\, G\, z_{ir}$ and consequently $\dfrac{\partial \frac{p_h^*}{p_s^*}}{\partial \omega_{ir}} = -\dfrac{1}{p_h^* \mathrm{D}_{p_h} E_h} \sum_{t=1,t\neq s}^{k} g_{ht} z_{t,ir}$ for $h \neq s$ and $h = 1,\ldots, k$, setting $r = h$, we get

$$\frac{\partial \frac{p_h^*}{p_s^*}}{\partial \omega_{ih}} = -\frac{1}{p_h^* \mathrm{D}_{p_h} E_h}\left(g_{hh} z_{h,ih} + \sum_{t=1,t\neq h,s}^{k} g_{ht} z_{t,ih}\right) < -\frac{g_{hh}}{p_h^* \mathrm{D}_{p_h} E_h} \sum_{t=1,t\neq s}^{k} z_{t,ih} < 0$$

since $\mathrm{D}_{p_h} E_h < 0$, $g_{hh} > g_{ht} > 0$ for $t \neq h$ and $z_{t,ih} = \dfrac{p_t^*}{p_s^*} \dfrac{\partial E_t}{\partial \omega_{ih}} > 0$ for $t \neq h$ and < 0 for $t = h$, with $\sum_{t=1,t\neq s}^{k} z_{t,ih} = \dfrac{1}{p_s^*} \sum_{t=1,t\neq s}^{k} p_t^* \dfrac{\partial E_t}{\partial \omega_{ih}} < 0$. Moreover, setting $r = s$, we get

$$\frac{\partial \frac{p_h^*}{p_s^*}}{\partial \omega_{is}} = -\frac{1}{p_h^* \mathrm{D}_{p_h} E_h} \sum_{t=1,t\neq s}^{k} g_{ht} z_{t,is} > 0$$

since $\mathrm{D}_{p_h} E_h < 0$, $g_{ht} > 0$ for $t \neq h$ and $z_{t,is} = \dfrac{p_t^*}{p_s^*} \dfrac{\partial E_t}{\partial \omega_{ih}} > 0$ for $t \neq s$. □

The assumptions in Proposition 12.14 are particularly strong, even though they are only sufficient conditions. Thus, it is possible to show examples, with not all the goods normal and gross substitutes, where the competitive prices are not indices of relative scarcity. Therefore, the statement (sometimes called a fundament of neoclassical theory) that competitive equilibrium prices are indices of scarcity, that is higher scarcity of a good implies *ceteris paribus* higher price, has no absolute value.

An interesting proposition looks at comparative statics of the social welfare function $W = \Sigma_{i=1}^{n} \dfrac{1}{\lambda_i} u_i(x_i)$ that is maximized by competitive equilibrium allocation (as shown in Proposition 11.16). We will now examine how social welfare, defined by this function, varies with changes in the quantity of resources. Since $W^*((\omega_i)_{i=1}^{n}) = \max_{(x,y)\in C_{FD}} \Sigma_{i=1}^{n} \dfrac{1}{\lambda_i} u_i(x_i) =$

$\Sigma_{i=1}^{n} \dfrac{u_i * (p^*, p^* \omega_i + \Sigma_{j=1}^{m} \theta_{ij} p^* y_j^*)}{\mathrm{D}_{m_i} u_i * (p^*, p^* \omega_i + \Sigma_{j=1}^{m} \theta_{ij} p^* y_j^*)}$, where ($x^*$, y^*, p^*) is a competitive equilibrium of $\mathcal{E} = (\langle X_i, \succsim_i \rangle, Y_j, \omega_i, \theta_{ij}, i = 1,\ldots, n, j = 1,\ldots, m)$, we need to determine derivatives $\mathrm{D}_{\omega_{ih}} W^*((\omega_i))$ in order to show how social welfare (represented by function $\Sigma_{i=1}^{n} \dfrac{1}{\lambda_i} u_i(x_i)$) varies when the quantity of resources changes.

Proposition 12.15 With respect to the social welfare function $W = \Sigma_{i=1}^{n} \dfrac{1}{\lambda_i} u_i(x_i)$, the change in social welfare determined by a change in the resources quantity is proportional to competitive equilibrium prices, that is $\mathrm{D}_{\omega_{ih}} W^*((\omega_i)) = p_h^*$ for every $h = 1,\ldots, k$ and $i = 1,\ldots, n$.

Proof. This proposition is a direct consequence of Proposition 8.11 and its comment. It is enough to realize that marginal rate of substitution among two goods on the curve of minimal resources to achieve utility u^* is equal to its price ratio in competitive equilibrium and that social welfare is expressed in the same accounting unit as prices. □

Proposition 12.15 states for the whole economy the Gossen-Menger *loss principle*, according to which the price (or value of a good) is equal to its marginal utility (which here is the derivative of the indirect social welfare function).

12.6 The core of an economy

Competitive equilibrium requires that agents base their choices on prices and that they are price-takers. Nevertheless, competitive equilibrium allocations are not only the result of this particular market regime, but are also a reflection of much stronger forces determined by the interaction among the agents, when they are very numerous. Core analysis sheds light on it.

Imagine an environment in which agents can exchange goods among themselves without constraints (only the agreement among the parties is necessary), that is without introducing a specific structure for exchange (like the one of competitive equilibrium, in which prices are equal for all agents). The result are exchanges (so allocations) with respect to which there is no possibility of improvement. Normally there are many allocations that have this property. Nevertheless, under very general conditions, these allocations not only include but also tend to coincide with competitive equilibrium allocation as the number of the agents increases. In other words, competitive equilibrium allocations are fundamental allocations for the economies composed of a large number of agents with freedom of exchange and production.

In the next part of this paragraph we will consider an economy (with free disposal) $\mathcal{E} = (\langle X_i, \succsim_i \rangle, Y, \omega_i, i \in N = \{1, \ldots, n\})$. We assume to have, for every $i \in N$, an endowment $\omega_i \in \mathbb{R}^k_+$, a consumption set $X_i = \mathbb{R}^k_+$ and a system of preferences $\langle X_i, \succsim_i \rangle$ that is regular, continuous, strictly convex and strongly monotonic. The production set $Y \subseteq \mathbb{R}^k$ (with $-\mathbb{R}^k_+ \subseteq Y$ by *free disposal*) has constant returns to scale and can be adopted by all consumers.[8] (We have a pure exchange economy if $Y = -\mathbb{R}^k_+$).

In order to continue the analysis, it is useful to introduce the following definitions.

Definition 12.5 (*Coalition*) A coalition is formed by a group of consumers. Every $S \subseteq N$, where $N = \{1, \ldots, n\}$, is a coalition.

Definition 12.6 (*Improvement upon an allocation*) A coalition can improve upon (or block) an allocation, if there are consumptions, feasible with the endowments of its members and with production set Y, preferred by all the consumers in the coalition to the consumptions of the examined allocation. Formally, a coalition $S \subseteq N$ can improve upon an allocation $(x_i)_{i \in N}$, if there are consumptions $(x_i')_{i \in S}$ such that

$$x_i' \succ_i x_i \quad \text{for every } i \in S$$

$$\Sigma_{i \in S}\ x_i' \in Y + \{\Sigma_{i \in S}\ \omega_i\}$$

Definition 12.7 (*Core of an economy*) The core of an economy is the set of all the allocations that cannot be improved upon by any coalition. Then, $x^* \in core(\mathcal{E})$ if, for every coalition $S \subseteq N$, $x_i' \succ_i x_i^*$ for every $i \in S$ implies $\Sigma_{i \in S}\, x_i' \notin Y + \{\Sigma_{i \in S}\, \omega_i\}$.

The representation of a pure exchange economy with two goods and two consumers by means of Edgeworth-Pareto box diagram, introduced in Paragraph 8.3, allows us not only to determine the core of this economy but also illustrate some of its properties that can be generalized. As shown in Figure 12.14, the *core* (sometimes called *contract curve*) is that part of the efficient allocations curve which is enclosed between the indifference curves going through the endowment. In fact, inefficient allocations can be improved upon by coalition {1, 2}; those on the south west of the indifference curve of the first consumer going through ω_1 could be improved by coalition {1}; and those on the north east of the indifference curve passing through ω_2 could be improved upon by coalition {2}.

[8] Excluding the possibility of production with increasing returns to scale, incompatible with competitive equilibrium which is confronted here with the core, the assumption that Y has constant returns to scale reflects the assumption that consumers have complete information about their exchange and production possibilities and are free to choose them. This implies that if $y \in Y$ then also $\lambda y \in Y$, for every non negative *integer* λ: constant returns to scale require that it holds also for every non negative *real* λ.

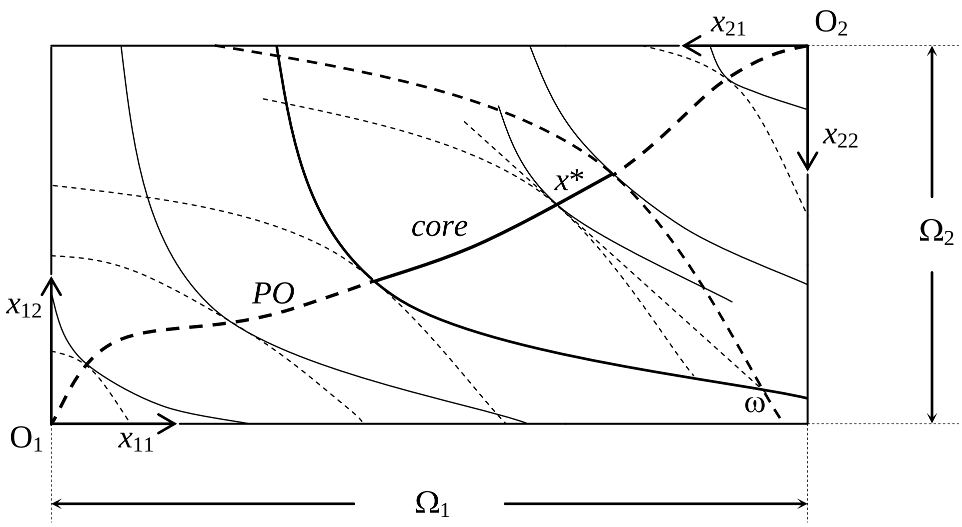

Figure 12.14

From Figure 12.14 we see that, on one hand, the core is composed of efficient allocations and, on the other hand, that every competitive equilibrium allocation belongs to core. In other terms, the core is a set that is included in the set of efficient allocations and includes the competitive equilibrium allocations. These two properties hold also for economies with more than two goods and two consumers, as shown in the following propositions.

Proposition 12.16 Every allocation that belongs to the core is efficient.[9]

Proof. We can easily prove an equivalent proposition according to which every feasible allocation that is not efficient does not belong to core. In fact, an allocation that is feasible and not efficient can be improved upon by coalition $S = N$ formed by all the consumers. □

Proposition 12.17 Every competitive equilibrium allocation belongs to the core.

Proof. With analogy to Proposition 11.12 (the first welfare theorem), we prove an equivalent proposition according to which an allocation cannot belong to a competitive equilibrium of an economy if it does not belong to its core. If an allocation $(x_i)_{i\in N}$ does not belong to the core, then there exists a coalition $S \subseteq N$ that can improve upon this allocation. That is there exist a $(x_i')_{i\in S}$ and a $y' \in Y$ such that $x_i' \succ_i x_i$ for every $i\in S$ and $\Sigma_{i\in S}\, x_i' \le y' + \Sigma_{i\in S}\omega_i$. As a consequence, for every competitive equilibrium vector of prices $p^* \in \mathbb{R}^k_+$, not only $\Sigma_{i\in S}\, p^*x_i' \le p^*y' + \Sigma_{i\in S}\, p^*\omega_i$ but also $\Sigma_{i\in S}\, p^*x_i' \le \Sigma_{i\in S}\, p^*\omega_i$

[9] We do not distinguish between weakly and strongly efficient allocations because, since the preferences are continuous and strongly monotone, every allocation that is weakly efficient is also strongly efficient (as shown in Proposition 8.3).

(recalling that $p^*y' \leq \max_{y\in Y} p^*y = 0$ by the assumption of constant returns to scale). Then, there is a $i\in S$ for which $p^*x_i' \leq p^*\omega_i$. Therefore, since $x_i' \succ_i x_i$, we get that x_i is not a preferred consumption in the budget set, that is $x_i \notin d_i(p^*)$. Thus, allocation $(x_i)_{i\in N}$ cannot belong to a competitive equilibrium. □

Proposition 12.17 is a significant extension of the first welfare theorem (introduced in Proposition 11.12), in the sense that a competitive equilibrium allocation cannot be improved upon not only unanimously but also by single consumers or groups of consumers, that is none of the consumers or group of the consumers would like to form a separate economy in order to achieve preferred consumptions. In other words, competitive equilibrium is robust also with respect to deviations of one consumer or a group of consumers.

The core depends on the possibility of forming coalitions. The more coalitions can be formed, the higher number of allocations is eliminated from the core. In this sense the core shrinks with an increase in the number of the possible coalitions. Having considered every subset of N as a possible coalition, it is an increase in the number of consumers n that gives rise to increase in the number of possible coalitions. Nevertheless, with an increase in n, also the geometric dimensions of the core increase. In fact, the core of an economy is a subset of the set of efficient allocation of which, in general, it has the same geometric dimensions, that are equal to $n-1$ (as shown at the end of Paragraph 8.2). Then, with an increase in n, on one hand, the core becomes a set with more dimensions, while, on the other hand, it is shrinking because of the increasing number of possible coalitions. In order to analyze this shrinking when n increases, we have to refer to a projection of core on a space with constant dimensions. This is obtained by introducing the notion of *replica* of an economy.

Let's imagine an economy in which the number of consumers is increased by duplication. That is new consumers that have the same preferences and endowments as existing consumers are added. Then, we can define the *type* of a consumer, characterized by a system of preferences and an endowment, such that the consumers of the same type have all the same preferences and endowment. In a *replica economy* the number of the consumers is increased by increasing the number of consumers of every type while leaving the number of the types unchanged. Then, if there are t types of consumers, denoting with T the set of types, the generic type of consumer is defined by $(\langle X_j, \succsim_j\rangle, \omega_j)$, where $j\in T = \{1, \dots, t\}$. In the r-th replica of the economy there are r consumers of any type, so the total number of consumers is $n = tr$.

For a replica economy introduced in this way the following proposition holds, according to which all the consumers of the same type have the same consumption in every allocation belonging to the core. This allows us to analyze the effect of increase in r on the core observing the projection of the *core* on the space of the type consumptions (where the consumption of the types of consumers are represented, so with only one consumer for each type). Then, denoting with $q \in R = \{1, \ldots, r\}$ the q-th cohort of consumers, the following proposition holds.

Proposition 12.18 In every allocation belonging to the core all the consumers of the same type are treated in the same way. That is, for every allocation $(x_{jq})_{j \in T,\ q \in R}$ (where x_{jq} is the consumption of the j-th type belonging to the q-th cohort) in the core of the r-th replica economy, we have $x_{jq} = x_{jq'}$ for every pair $q,q' \in R$ and for every $j \in T$.

Proof. We prove that every allocation for which the equal treatment property does not hold, does not belong to the core. Then we consider whatever allocation in which there is, for every type, a consumer that is treated not better than any other consumer of the same type and, for at least one type, there is a consumer that is treated worse than any other consumer of the same type. Setting, without loss of generality, all those agents in the r-th cohort, we have $x_{jr} \precsim_j x_{jq}$ for every pair with $j \in T$ and $q \in R$ and $x_{j'r} \prec_{j'} x_{j'q'}$ for at least one pair with $j' \in T$ and $q' \in R$. Denoting with $\overline{x}_j = \frac{1}{r} \Sigma_{q \in R}\, x_{jq}$ the average consumption of j-th type agents, by convexity of preferences we get that $x_{jr} \precsim_j \overline{x}_j$ for every $j \in T$ and $x_{j'r} \prec_{j'} \overline{x}_{j'}$ for at least one $j' \in T$. However, the coalition formed by all the consumers of the r-th cohort can let each consumer receive the average consumption of his type because it has the quantity $\Sigma_{j \in T}\, \omega_j$ of goods and can use technology Y. In fact, since allocation $(x_{jq})_{j \in T,\ q \in R}$ is feasible, we have $\Sigma_{j \in T}\, \Sigma_{q \in R}\, x_{jq} = r\, \Sigma_{j \in T}\, \overline{x}_j = y + r\, \Sigma_{j \in T}\, \omega_j$, where $y \in Y$. Therefore, since $\frac{1}{r} y \in Y$ because Y has constant returns to scale, we have $\Sigma_{j \in T}\, \overline{x}_j \in Y + \{\Sigma_{j \in T}\, \omega_j\}$, so it is possible for the coalition of all the r-th cohort consumers to assign to each type his average consumption. Moreover, it is possible to assign to each type a consumption preferred to the one of the examined allocation $(x_{jr})_{j \in T}$. In fact, by continuity of preferences, there exists an $\varepsilon \neq 0$, with $\varepsilon \in \mathbb{R}^k_+$, for which $\overline{x}_{j'} - \varepsilon \succ_{j'} x_{j'r}$ and by strong monotonicity of preferences $\overline{x}_j + \frac{1}{t-1} \varepsilon \succ_j \overline{x}_j \succsim_j x_{jr}$ for every $j \neq j'$. As a result, all the consumers of the same type have, in every allocation in the core, indifferent consumptions, that is $x_{jq} \sim_j x_{jq'}$ for every $j \in T$ and every pair $q,q' \in R$. In order to prove that they are not only indifferent but also equal, that is $x_{jq} = x_{jq'}$ for every $j \in T$ and every pair $q,q' \in R$, we consider an allocation in which there is a pair of consumers of the same type for whom $x_{jq} \neq x_{jq'}$ and $x_{jq} \sim_j x_{jq'}$. This allocation is not efficient: the situation of this pair of consumers can be improved without making anybody worse off since $\frac{1}{2}(x_{jq} + x_{jq'}) \succ_j x_{jq} \sim_j x_{jq'}$. So by Proposition 12.16 every allocation $(x_{jq})_{j \in T,\ q \in R}$ with $x_{jq} \neq x_{jq'}$ for some $j \in T$ and $q,q' \in R$ does not belong to the core. □

The fact that all the consumers of the same type have the same consumption, if the allocation belongs to the core, allows us to analyze how the core changes when replica r increases considering only the types of the consumers and not all of them. It is possible to examine only allocations $(x_j)_{j\in T}$ of types (remembering that in the core we have $x_{jq} = x_j$ for every $q\in R$ and $j\in T$). We denote with $C_r(\mathcal{E})$ the projection of the $core(\mathcal{E})$ on the allocations space of types when the number of consumers is $n = tr$. We get, on the one hand, that $C_r(\mathcal{E}) \subseteq C_{r-1}(\mathcal{E})$ (because every allocation of the types that does not belong $C_{r-1}(\mathcal{E})$ does not belong also to $C_r(\mathcal{E})$, since the coalitions available for replica r–1 are also available for replica r) and, on the other hand, we get by Proposition 12.17 that competitive equilibrium allocations belong to $C_r(\mathcal{E})$ for every value of r (because they belong to the $core(\mathcal{E})$ for every value of n). The following proposition proves that with increases in r every allocation that is not a competitive equilibrium allocation is excluded from $C_r(\mathcal{E})$, that is $core(\mathcal{E})$ converges, with increases in r, towards the set of competitive equilibrium allocations.

Proposition 12.19 If an allocation $(x_j)_{j\in T}$ is not a competitive equilibrium allocation, then there exists sufficiently large replica r such that $(x_j)_{j\in T} \notin C_r(\mathcal{E})$.

Proof. This proposition is proved, assuming, apart from what has already been indicated in the beginning of this paragraph, that $x_j >> 0$ for every $j\in T$ and that the preferences of the consumers can be represented with differentiable utility functions. If $(x_j)_{j\in T}$ is not a feasible allocation or when it is feasible but not efficient, then, by Proposition 12.16, it does not belong to $C_r(\mathcal{E})$ whatever the value of r is. So let's consider an efficient allocation $(x_j)_{j\in T}$ belonging to $C_r(\mathcal{E})$ for some r (that is, $(x_{jq})_{j\in T,\ q\in R}$ is an efficient allocation belonging to the core, so with $x_{jq} = x_j$ for every $q\in R$ and $j\in T$). Since it is efficient, by second welfare theorem (Proposition 11.8) there exists a competitive equilibrium $((x_j)_{j\in T}, y, p)$ for endowments $(\omega_j')_{j\in T} = (x_j)_{j\in T}$ (recall that $py = 0$, that is the profit is zero in competitive equilibrium because the production set Y has constant returns to scale). If $((x_j)_{j\in T}, y, p)$ is not a competitive equilibrium for endowments $(\omega_j)_{j\in T}$, then (recalling that $py = 0$) we cannot have $px_j \le p\omega_j$ for every j, so there is at least one consumer, denoted with t, for whom $p\omega_t' = px_t > p\omega_t$. Let's consider the coalition formed by all the consumers except for the r-th consumer of type t. This coalition can distribute to its members the quantity of goods $y + r\Sigma_{j\in T}\omega_j - \omega_t$, for example assigning $x_{jq}' = x_j + \dfrac{1}{r-1+r(t-1)}(x_t - \omega_t)$. In fact, $\Sigma_{j\in T\setminus\{t\}}\Sigma_{q\in R}x_{jq}' + \Sigma_{q\in R\setminus\{r\}}x_{tq}' = r\Sigma_{j\in T\setminus\{t\}}x_j + \dfrac{r(t-1)}{r-1+r(t-1)}(x_t - \omega_t) + (r-1)x_t + \dfrac{r-1}{r-1+r(t-1)}(x_t - \omega_t) = r\Sigma_{j\in T}\, x_j - \omega_t = y + r\Sigma_{j\in T}\,\omega_j - \omega_t$. At this point, we need to prove that $x_{jq}' \succ_j x_j$ for every $j\in T$ and for every sufficiently large r.

In other words, if the economy is composed of many consumers this coalition can block a non competitive allocation. This result is obtained taking into account that with increases in r the variation $x_{jq}' - x_j$ (that is a difference between the consumption that the coalition can assign and the one of the examined allocation) becomes smaller and that the preferences of the consumers can be represented, by assumption, with differentiable utility functions. We find out that, considering the first order terms of the Taylor series expansion of the utility function and recalling that $((x_j)_{j\in T}, y, p)$ is a competitive equilibrium so $\mathrm{D}_{x_j} u_j(x_j) = \lambda_j\, p$ (with $\lambda_j > 0$),

$$u_j(x_{jq}') - u_j(x_j) \simeq \mathrm{D}_{x_j} u_j(x_j)\,(x_{jq}' - x_j) = \frac{\lambda_j}{r-1+r(t-1)}\, p\,(x_t - \omega_t) > 0$$

for every consumer belonging to the coalition. □

Proposition 12.19 can be seen as a generalization of the second welfare theorem, just like Proposition 12.17 was seen as an extension of the first one. It demonstrates how competitive equilibrium allocations are those to which the economies, composed of large number of consumers, converge if there is freedom to carry out exchanges and production and there is enough information on these possibilities.[10]

12.7 Equilibrium, time and uncertainty

In the beginning of Chapter 3, we introduced goods and qualified them according to their physical features (for example, wheat and steel), place of delivery (for example, Chicago or Milan), delivery date (for example now or in three months) and state of the nature, so whether a certain event occurs or not (for example, whether there is hail or not). In this paragraph we will concentrate on the last two characteristics. (Terms “date”, “time” and “instant” will be used as synonyms meaning a point in the time axis).

As example of the first characteristic notice that production choices involve time directly, because the inputs have to be used normally before (in any case, not after) the outputs emerge. Therefore, inputs are qualified with an earlier (or not later) date than corresponding outputs. For the second characteristic, insurance contracts are a typical example. Contracts are based on state of the nature and harmful event gives right to contracted payment.

Moreover, there are contracts on exchanges that are drawn up in different dates and indicate the delivery of goods in the same date of the contract or later.

[10] For the model with a continuum of agents, introduced in paragraph 11.13, there is an important Aumann’s theorem which states that the core coincides with the set of competitive equilibrium allocations under the same assumptions as for the existence of a competitive equilibrium,

Therefore, for every good (wheat or steel, available in a certain place) prices can differ with respect to contract date, delivery date and state of the nature that determines the right to obtain it. In the remaining of this paragraph we assume that every good is delivered in exchange for the numeraire good (which is one of the goods in the economy with nominal price that is always positive, we assume it is the good indexed with 1)[11]. With this assumption, the price of the good in question, with the numeraire as the accounting unit, can be denoted with symbol $p_h(t,b,c,s)$, where h denotes *the type of the good* (wheat or steel available in a certain place), t the *contract date*, b (with $b \geq t$) the *payment date* in numeraire, c (with $c \geq t$) the *delivery date* and s the *event* that determines the right for the purchaser to obtain the good.[12]

With respect to contract date, two main types of the economies can be defined. The first one considers economies in which there is a unique time in which the exchange contracts are fixed and payments carried out. The corresponding equilibrium is called *intertemporal equilibrium.* Indicating with the number zero the contract and payment date, the competitive equilibrium concerns prices that can be denoted with $p_h(c,s)$, where $c \geq 0$ (since the prices $p_h(t,b,c,s)$ are in these economies $p_h(0,0,c,s)$ and, thus, can be denoted, for simplicity, as $p_h(c,s)$). The other type looks at *sequential economies* in which there is a sequence of times in which the contracts are established, that is with $t = 0, 1, 2, \ldots$ An equilibrium, called a *temporary equilibrium*, corresponds to each of those instants.[13] These two types of the economies determine in general different competitive allocations, also if the "data" that define the economies are substantially the same (however,

[11] It is natural to assume money as numeraire, so that the indicated prices are monetary prices. Nevertheless, introduction of money can create problems for general equilibrium analysis. For example, if money is qualified as a durable good that agents hold only because it can be used as (non exclusive) payment device, then its presence (with positive prices) is not justified in the intertemporal equilibrium (that will be defined shortly). In fact, in the intertemporal representation of an economy, there is only one date for the payments and nobody wants a positive quantity of costly, useless and unusable money in his bundle of goods. Also in the temporary equilibrium analysis, if time horizon is finite, introduction of money is problematic (if its value is zero in the last period then proceeding by backward induction it is also equal to zero in earlier periods). Therefore, we must assume specifically that money is better means of payment than other goods. We can motivate the use of money for example by relaxing assumption that exchanges are not costly and introducing *transaction costs* that are higher for non monetary exchanges.

[12] In what follows, when we discuss the relationship between intertemporal and sequential equilibrium, we adopt a notation that allows us to keep in mind that also payment for the good (and not only its delivery) can be contingent on some event. Denoting the date-event pair of the payment for the good with s_r^b (that is the r-th event in time b) and with s_r^c the date-event pair of the delivery of the good, the corresponding price is $p_{h,s_r^b,s_r^c}(t)$, where t is the contract date.

[13] If the sequence of temporary equilibria is composed of equilibria that are all the same, then equilibrium is defined as *stationary*, just like anything that remains unchanged in time is defined.

consumption and production allocations coincide under the conditions stated, in the end of this paragraph, in Proposition 12.21).

Equilibrium and time: interest rates. The comparison between the *spot prices* (i.e. with $t = b = c$) and the *forward prices* (i.e. with $t = b < c$) allows us to define *interest rates* (both for intertemporal and temporary equilibrium). Let's consider the intertemporal equilibrium (in which all the contracts are formed at date 0 with payments at date 0) and, in which, for now, the contracts are not contingent on the events (i.e. with delivery of the good no matter what the state of the nature is, therefore with prices $p_h(c, S_c)$, where S_c indicates the sets of all possible states of the nature at time c, for convenience $p_h(c)$).

Definition 12.8 The *interest rate* for the h-th good for period between 0 and c (with $c > 0$) is the ratio

$$i_h(c) = \frac{p_h(0) - p_h(c)}{p_h(c)},$$

where $p_h(0)$ and $p_h(c)$ are, respectively, the spot price of the h-th good (that is contract, payment and delivery occur at date 0) and its forward price (that is delivery is postponed with respect to contract and payment date, so with some form of *credit*). Usually, we have $p_h(c) < p_h(0)$ (in fact, with postponed delivery, we pay in advance with respect to the delivery of the good). Applying this definition to the numeraire good, for which by definition $p_1(0) = 1$, we get

$$p_1(c) = \frac{1}{1 + i_1(c)},$$

that is, $p_1(c)$, which is the *present value* of one unit of numeraire good available at time c, defines the *discount factor* of the numeraire for the period between 0 and c. We note that interest rates are in general different for different goods (apart from being different for periods of different length).

Let's assume that there is a sequence of temporary equilibria. When we examine the temporary equilibrium at date 0, besides spot and forward prices there can also be *future prices*. Future prices consider the contracts in which the delivery of the good and payment are simultaneous, but postponed with respect to contract date (that is, $b = c > t$). For example, future price established at time 0 for the sale of wheat with delivery in six months and payment in six months. Then, considering also the payment date, the forward price of the h-th good can be denoted with $p_h(0,c)$ (rather than as we did before $p_h(c)$) and the future price with $p_h(c,c)$. Competitive forward and future prices are not independent. In fact, an agent can acquire the h-th good with delivery at time c using two diverse exchange ways. For example, he can buy the good forward: have a unit of good in time c paying at time 0 price $p_h(0,c)$. Otherwise, he can buy the good with the future contract, that gives him the right to get a unit of the good at time c, paying at time c price $p_h(c,c)$, and with a forward contract he buys the amount of numeraire he needs at time c paying at time 0 the price $p_1(0,c)$ for every unit of the numeraire. Following the first way, at time 0 the price of good available at time c is $p_h(0,c)$. Following the second way, at time 0 the price that corresponds to the quantity of the numeraire to be paid for to have at time c the quantity of the numeraire established in the future contract is equal to the product $p_h(c,c)\ p_1(0,c)$. Competitive market makes those two prices equivalent (otherwise, one could achieve infinite profit without any risk, buying at price $\min\{p_h(0,c), p_h(c,c)\ p_1(0,c)\}$ and selling at $\max\{p_h(0,c), p_h(c,c)\ p_1(0,c)\}$). This *no-arbitrage condition* requires then

$$p_h(0,c) = p_h(c,c)\ p_1(0,c)$$

This condition has two implications. The first one reduces the number of markets present in the temporary equilibrium: if there are future markets, then it is sufficient that forward markets consider only the numeraire good (on these markets we deal with bonds without risk and without coupons, that is of the *zero-coupon* type, often denoted in general equilibrium literature as *Arrow-Debreu bonds*). The other implication concerns the interest rates. From the definition $i_h(0,c) = \dfrac{p_h(0,0) - p_h(0,c)}{p_h(0,c)}$ and the no-arbitrage condition $p_h(0,c) = p_h(c,c)\ p_1(0,c)$ we obtain

$$i_h(0,c) = \frac{1}{p_1(0,c)} \frac{p_h(0,0)}{p_h(c,c)} - 1,$$

so we prove that the own-interest rates of the different goods are linked through ratios between spot and future prices. We get the following proposition.

Proposition 12.20 If $p_h(c,c) = p_h(0,0)$ for all goods, that is, future and spot prices are equal, then interest rates are equal for all the goods and coincide with the interest rate of numeraire good for the same period.

Moreover, future and spot prices are equal if all the agents have *stationary prices expectations,* which is a sufficient condition for the existence of a unique interest rate for all goods.

This implication is connected with *speculation*. Let's assume that there are risk-neutral or risk-loving agents who speculate. Suppose that at time 0 one speculator expects a spot price at time $c > 0$ that will be different than the corresponding future price, that is $p_h(c,c,c)^e \neq p_h(0,c,c)$ (where $p_h(0,c,c)$ is the above indicated price with symbol $p_h(c,c)$ and $p_h(c,c,c)^e$ is the expected spot price for the temporary equilibrium at time c). Then, it is convenient for him to speculate, buying (selling) the good with the future contract if $p_h(c,c,c)^e$ is larger (smaller) than $p_h(0,c,c)$, selling (buying) the good in the spot market at time c and achieving in such a way a profit. (Note that this type of arbitrage differs from the one, examined before, that was governed by condition $p_h(0,0,c) = p_h(0,c,c)\ p_1(0,0,c)$. Previously, the profit achieved with arbitrage was certain, while now it is uncertain since it is subject to the realization of the expectations). If all the agents (speculating and not speculating) have stationary price expectations, that is $p_h(c,c,c)^e =$

$p_h(0,0,0)$, and it were $p_h(0,0,0) \neq p_h(0,c,c)$, then all the speculating agents operating in this market would act in the same direction, that is all buy or all sell and the equilibrium does not exist. This, consequently, implies the speculative no-arbitrage condition $p_h(0,0,0) = p_h(0,c,c)$ (that is $p_h(0,0) = p_h(c,c)$, if we neglect the index for the contract date that is 0).

Equilibrium and uncertainty: contingent prices. We will now study dependence of prices on states of the nature in intertemporal equilibrium analysis. Therefore, we denote prices with $p_h(c,s)$. These prices, relative to an event, are called *contingent* just like the corresponding goods and markets.[14] The buyer of a contingent good pays at time 0 price $p_h(c,s)$ and he receives the good at time c if event s occurs and he receives nothing if this event does not occur (the price of the good independent of the state of nature, so the price to receive the good with certainty, is $p_h(c, S_c) = \sum_{s\in S_c} p_h(c,s)$, where S_c denotes the set of all possible states of nature at time c). Insurance and many financial contracts are of this type. For example, as shown in Paragraph 7.7, in an insurance contract, we pay a price (called *premium*) in order to receive at some future point in time a sum of money if a harmful event stated in the contract occurs and nothing if it does not occur.

We assume that the time horizon is composed of a finite number of points, that is $c \in \{0, 1,\ldots, T\}$ (where T is the last instant for the delivery of goods). We denote the set of possible states of the nature at time c with S_c. As the time passes by more information is produced and so the set of states of nature is refined. Therefore, the number of elements in sets S_c is non decreasing in c and so the set $S_T = \{ s_1^T,\ldots, s_{R_T}^T \}$ has the largest number of elements. We assume that this number, R_T, is finite. Then, starting from S_T, since S_T is a partition of $S_{T-1} = \{ s_1^{T-1},\ldots, s_{R_{T-1}}^{T-1} \}$, we have that every element s_r^{T-1} of S_{T-1} is a subset of S_T; then, since S_{T-1} is a partition of $S_{T-2} = \{ s_1^{T-2},\ldots, s_{R_{T-2}}^{T-2} \}$ we get that every s_r^{T-2} is a subset of S_{T-1}; and so on. Then, we obtain $S_0 = \{ s_1^0 \}$ with $s_1^0 = S_T$, because in the first period no information is available. Therefore, $1 = R_0 \leq R_1 \leq\ldots\leq R_{T-1} \leq R_T$.

Finally, we assume that sets S_c, for $c = 1,\ldots, T$, are equal for all the agents, that is, that there is common information (no informational asymmetries).

[14] The definitions of state of the nature and event (that is a set of states of nature) are stated in the beginning of Chapter 7. Notice that events must be objectively observed. For example, one cannot insure against sadness (assuming that the sadness cannot be objectively observed because the agent can only pretend to be sad).

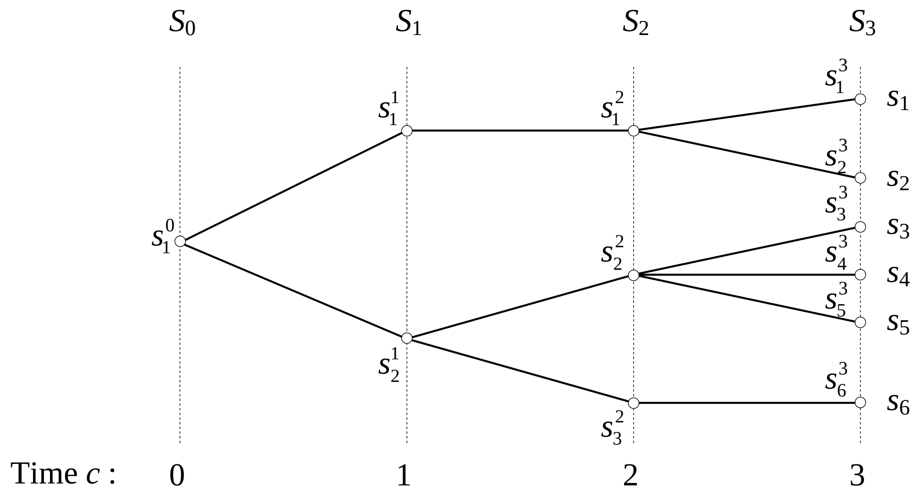

Figure 12.15

The increasing refinement of the sets of states of nature S_c is exemplified in Figure 12.15 by a *tree of events*, where $T = 3$; $S_0 = \{ s_1^0 \}$ with $s_1^0 = \{s_1, s_2, s_3, s_4, s_5, s_6\}$; $S_1 = \{ s_1^1, s_2^1 \}$ with $s_1^1 = \{s_1, s_2\}$ and $s_2^1 = \{s_3, s_4, s_5, s_6\}$; $S_2 = \{ s_1^2, s_2^2, s_3^2 \}$ with $s_1^2 = \{s_1, s_2\}$, $s_2^2 = \{s_3, s_4, s_5\}$ and $s_3^2 = \{s_6\}$; and $S_3 = \{ s_1^3, \ldots, s_6^3 \}$ with $s_r^3 = s_r$ for $r = 1,\ldots,6$.

With regard to every delivery date c, there are R_c prices $p_h(c,s)$ (with $s \in S_c$) for every good. Therefore, for every good, there are overall (considering all the possible delivery dates) $R = \sum_{c=0}^{T} R_c$ prices. The intertemporal competitive equilibrium, with *complete markets*, determines all of them for all the goods. Recalling our description (until the preceding paragraph) of the general competitive equilibrium, in which we listed k goods, since every good is now qualified by a pair $(h,\ s_r^c)$, the total number of goods (defined also by delivery date and the state of nature on the top of physical characteristics and place of delivery) is equal to $k = KR$, where K is the number of goods determined by their physical characteristic and place of delivery.

If the sequential equilibrium is taken under consideration, observing the temporary equilibrium at time 0 and the set of contingent markets and corresponding future markets, we get the following no-arbitrage condition (analogous to the one that we have already seen for non contingent prices)

$$p_h(0,c,s) = p_h(c,c,s)\, p_1(0,c,s),$$

where $p_h(c,c,s)$ is the future price of the h-th good contingent on event s at time c. Notice that contingent forward markets for only the numeraire good are necessary, that is *Arrow-Debreu* contingent bonds for every event $s \in \bigcup_{c=1}^{T} S_c$. In other words, while in the case without uncertainty we need an *Arrow-Debreu* bond for every possible future date (therefore, T bonds), in case of uncertainty there are R–1 bonds (i.e. R_c bonds for every $c = 1,\ldots, T$). Contingent bond gives the right to the delivery of a unit of numeraire at date

c if and only if state of the nature s_r^c is revealed, where $r = 1,\dots R_c$ and $c = 1,\dots, T$.

Note that by including uncertainty we do not change general competitive equilibrium representation formed in the preceding chapter and preceding parts of this chapter. We account for uncertainty only by a semantic trick that defines the goods not only in relation to their physical characteristics and their localization but also with respect to all the possible states of the nature. Therefore, it is not necessary to specify agents' preferences over the sets of acts (or lotteries), following the analysis from Chapter 7. In particular it is not necessary to assume that expected utility theory holds. It is sufficient that agents have systems of preferences $\langle X_i, \succsim_i\rangle$ (exhibiting properties required in Paragraphs 11.4 and 11.6), in which consumption sets X_i are composed of bundles of contingent goods and firms have production sets Y_j composed of contingent productions. Nevertheless, the number of implied goods (for every possible delivery place and date and state of the nature), even if it is finite, is enormous and it is doubtful whether general equilibrium representation that determines competitive prices of all these goods is a rational reconstruction of reality. General equilibrium with incomplete markets (that will be presented in Paragraph 12.9) is a response to this observation. With incomplete markets we assume that the market features lower than total number of goods and so competitive prices are determined only for a part of the goods.

In the last part of this paragraph we compare intertemporal and sequential equilibrium and characterize conditions under which consumption and production allocations determined by intertemporal equilibrium are equal to the ones determined by the sequence of temporary equilibria.[15] Let's consider an intertemporal economy $\mathcal{E} = (\langle X_i, \succsim_i\rangle, Y_j, \omega_i, \theta_{i,j}, i = 1,\dots, n, j = 1,\dots, m)$ and introduce a sequential economy defined by temporary economies, one for every node of event tree, $\mathcal{E}(s_r^t) = (\langle X_i(s_r^t), \succsim_i^{s_r^t}\rangle, Y_j(s_r^t), \omega_i(s_r^t), \theta_{i,j}(s_r^t), i = 1,\dots, n, j = 1,\dots, m)$, with $t = 0, 1,\dots, T$. These economies are defined and connected with each other through the following relationships.

[15] We do not consider here the possibility of *bankruptcy*, so possibility that some agents may not honor the obligation they took on. This problem occurs in forward contracts (in spot settlement nobody has reason to sign a contract that he cannot honor, because he gets nothing in return). An agent can sell a good forward, collecting immediately the price and refusing to deliver the good later on. To avoid such possibility, economy has to employ penalties to make such behavior not profitable for agents. This solution can be applied in intertemporal equilibrium with complete markets even if there is uncertainty. On the contrary, in case of sequential equilibrium with incomplete markets, uncertainty can generate unintentional bankruptcy and this is a serious problem. An agent could not to honor a forward contract because it became too onerous and this would lower agents' availability to draw up such forward contracts. For this reason there are special institutions in the markets that provide guarantee against the risk of non-fulfilment.

With respect to the intertemporal economy, introducing the tree of events and denote its nodes with the set $CS = \cup_{c=0}^{T} S_c$ (so that every element $s_r^c \in CS$ characterizes a date-event pair), consumptions and productions are, respectively, points $x_i = (x_{i,h,s_r^c})_{h\in H, s_r^c\in CS}$ and $y_j = (y_{j,h,s_r^c})_{h\in H, s_r^c\in CS}$ of the sets $X_i \subset \mathbb{R}^K \times CS$ and $Y_i \subset \mathbb{R}^K \times CS$, where $H = \{1,\dots, K\}$ is the set of the goods distinguished on the base of physical characteristics and delivery place. Then, $\omega_i = (\omega_{i,h,s_r^c})_{h\in H, s_r^c\in CS} \in X_i$ and $\theta_{i,j} \in[0,1]$ and $\sum_{i=1}^{n} \theta_{i,j} = 1$ for every $j = 1,\dots, m$ and $s_r^c \in CS$. The prices are represented by a point $p = (p_{h,s_r^c})_{h\in H, s_r^c\in CS} \in \mathbb{R}^K \times CS$, where the spot price of the numeraire is equal to 1, that is $p_{1,s_1^0} = 1$. Budget set of the i-th consumer is, as usual, $B_i(p) = \{x_i\in X_i : px_i \leq p\omega_i + \sum_{j=1}^{m} \theta_{i,j} \max_{y_j\in Y_j} py_j \}$, where $px_i = \sum_{h\in H, s_r^c\in CS} p_{h,s_r^c} x_{i,h,s_r^c}$ and by analogy for $p\omega_i$ and py_j. Production choices are represented by supply functions $s_j(p) = \arg\max_{y_j\in Y_j} py_j$ and consumption choices by demand functions $d_i(p) = \{x_i\in B_i(p)\text{: } x_i \succsim_i x_i' \text{ for every } x_i'\in B_i(p)\}$. The competitive intertemporal equilibrium $((x_i^*)_{i=1}^{n}, (y_j^*)_{j=1}^{m}, p^*)$ is defined by conditions $x_i^*\in d_i(p^*)$ for every $i = 1,\dots, n$, $y_j^*\in s_j(p^*)$ for every $j = 1,\dots, m$ and $\sum_{i=1}^{n} x_i^* \leq \sum_{i=1}^{n} \omega_i + \sum_{j=1}^{m} y_j^*$.

With respect to sequential economy, introducing for every $s_r^t \in CS$ the subtree of events that originates at s_r^t and denoting its nodes with set $CS(s_r^t)$ (for which every element $s_r^b \in CS(s_r^t)$ or $s_r^c \in CS(s_r^t)$, with $b \geq t$ and $c \geq t$, characterizes a date-event pair (c,s) still possible at date-event s_r^t), exchange choices of the i-th consumer are points $z_i(s_r^t) = (z_{i,h,s_r^b,s_r^c}(s_r^t))_{h\in H; s_r^b,s_r^c\in CS(s_r^t)} \in \mathbb{R}^K \times (CS(s_r^t))^2$, where $z_{i,h,s_r^b,s_r^c}(s_r^t)$ denotes, for the i-th consumer, sale or purchase (a purchase if positive and selling if negative) of the h-th good chosen at time-event s_r^t with payment (in numeraire) at time-event $s_r^b \in CS(s_r^t)$ and delivery of the good at date-event $s_r^c \in CS(s_r^t)$ (with $b \geq t$ and $c \geq t$). The consumers can also buy and sell the property shares in the firms, paying (in numeraire) at date-event $s_r^b \in CS(s_r^t)$ and delivery at date-event $s_r^c \in CS(s_r^t)$, with $b \geq t$ and $c \geq t$, and buyers have the right to the corresponding flow of profit from the stated delivery date. Therefore, they can choose exchanges of shares represented by points $\zeta_{i,j}(s_r^t) = (\zeta_{i,j,s_r^b,s_r^c}(s_r^t))_{s_r^b,s_r^c\in CS(s_r^t)}$, where $\zeta_{i,j,s_r^b,s_r^c}(s_r^t)$ denotes, for the i-th consumer, the acquired share (if positive, or sold if negative) in the j-th company (with $j\in J$) at time-event s_r^t, with payment (in numeraire) at date-event $s_r^b \in CS(s_r^t)$ and delivery at date-event $s_r^c \in CS(s_r^t)$ (with $b \geq t$ and $c \geq t$). The choices of the j-th firm regard points $y_j(s_r^t) = (y_{j,h,s_r^b,s_r^c}(s_r^t))_{h\in H;s_r^b,s_r^c\in CS(s_r^t)}$ $\in \mathbb{R}^K \times (CS(s_r^t))^2$, with $\sum_{s_r^b\in CS(s_r^t)}(y_{j,h,s_r^b,s_r^c}(s_r^t))_{h\in H;s_r^c\in CS(s_r^t)} \in Y_j(s_r^t)$, where $Y_j(s_r^t) \subset \mathbb{R}^K \times CS(s_r^t)$ is the production set of the j-th company at date-event s_r^t. The prices of the goods are represented, in every $s_r^t \in CS$, by point $p(s_r^t)$ = $(p_{h,s_r^b,s_r^c}(s_r^t))_{h\in H;s_r^b,s_r^c\in CS(s_r^t)} \in \mathbb{R}_+^K \times (CS(s_r^t))^2$, with the price of the numeraire equal to 1 if the payment is foreseen for the same date-event as delivery, that is with $p_{1,s_r^b,s_r^c}(s_r^t)$ = 1 if $s_r^b = s_r^c$. The prices of the firms are represented by points $q_j(s_r^t)$ = $(q_{j,s_r^b,s_r^c}(s_r^t))_{s_r^b,s_r^c\in CS(s_r^t)} \in (CS(s_r^t))^2$ for every $s_r^t \in CS$ and $j = 1,\ldots, m$. Consider the branch of the events that precede s_r^t (that is, the branch that connects s_1^0 with s_r^t in the tree of the events) and denote its nodes (excluding s_r^t) with $CA(s_r^t)$. The endowment of the i-th individual, which is the result of exchanges carried out in the past, is at date-event s_r^t represented by $\omega_i(s_r^t) = (\omega_{i,h,s_r^c}(s_r^t))_{h\in H;s_r^c\in CS(s_r^t)}$, with $\omega_{i,h,s_r^c}(s_r^t)$ = $\omega_{i,h,s_r^c} + \sum_{s_r^a\in CA(s_r^t);s_r^b\in CS} z_{i,h,s_r^b,s_r^c}(s_r^a)$ for the goods $h = 2,\ldots, H$ and $\omega_{i,1,s_r^c}(s_r^t) =$ $\omega_{i,1,s_r^c} + \sum_{s_r^a\in CA(s_r^t);s_r^b\in CS} z_{i,1,s_r^b,s_r^c}(s_r^a) - \sum_{h\in H;s_r^a\in CA(s_r^t);s_r^{c'}\in CS} p_{h,s_r^c,s_r^{c'}}(s_r^a) z_{i,h,s_r^c,s_r^{c'}}(s_r^a) -$ $\sum_{j\in J;s_r^a\in CA(s_r^t);s_r^{c'}\in CS} q_{j,s_r^c,s_r^{c'}}(s_r^a)\zeta_{i,j,s_r^c,s_r^{c'}}(s_r^a)$ for the numeraire, where ω_{i,h,s_r^c} is the original endowment of the h-th good available in s_r^c, i.e. the endowment that would be available in the absence of sales and purchases in preceding times, and $z_{i,h,s_r^b,s_r^c}(s_r^a)$ is a sale or purchase contracted in preceding date-events for the delivery in date-event $s_r^c \in CS(s_r^t)$. For the numeraire (the good indexed with 1) we also need to keep track of the payments and the receipts resulting from sales and purchases (also of Arrow-Debreu bonds)[16] contracted at preceding date-events and expiring at date-events $s_r^c \in CS(s_r^t)$ (while $s_r^{c'}$ is date-event of the delivery of the good or share of a firm). By analogy, denoting with $s_{r*}^{t-1} \in CA(s_r^t)$ the date-event that immediately precedes s_r^t, the endowment of firms' shares of the i-th consumer is represented at date-event s_r^t by $\theta_{i,j}(s_r^t)$ = $\zeta_{i,j}(s_r^t) + (\theta_{i,j,s_r^c}(s_r^t))_{s_r^c\in CS(s_r^t)}$, where $\theta_{i,j,s_r^c}(s_r^t)$ = $\theta_{i,j,s_r^c}(s_{r*}^{t-1}) + \sum_{s_r^a\in CA(s_r^t);s_r^b\in CS}\zeta_{i,j,s_r^b,s_r^c}(s_r^a)$, with $\theta_{i,j,s_r^c}(s_1^0)$ = θ_{i,j,s_r^c}, which is the endowment without sales and purchases, and $\theta_{i,j,s_r^c}(s_r^t) \in [0,1]$, with $\sum_{i=1}^n \theta_{i,j,s_r^c}(s_r^t)$ = 1, for every $j = 1,\ldots, m$, $s_r^c \in CS(s_r^t)$

[16] Therefore, $z_{i,1,s_r^b,s_r^c}(s_r^a)$ is encashment (if positive) or payment (if negative) of the Arrow-Debreu bonds (bought and sold by the i-th agent at date-event s_r^a) expiring at date-event s_r^c.

and $s_r^t \in CS$. For the i-th consumer the intertemporal consumption plan in s_r^t for date-events of the subtree $CS(s_r^t)$ is defined by the point $\omega_i(s_r^t) + \sum_{s_r^b \in CS(s_r^t)} z_i(s_r^t) = x_i(s_r^t) \in X_i(s_r^t) \subset \mathbb{R}^K \times CS(s_r^t)$. Using symbols we introduced, the budget set is $B_i(p(s_r^t), (\pi_j^*(s_r^t))_{j=1}^m) = \{z_i(s_r^t) \in \mathbb{R}^K \times (CS(s_r^t))^2$: $\omega_i(s_r^t) + \sum_{s_r^b \in CS(s_r^t)} z_i(s_r^t) = x_i(s_r^t) \in X_i(s_r^t)$ and $p(s_r^t)\, z_i(s_r^t) + \sum_{j=1}^m q_j(s_r^t)\zeta_{i,j}(s_r^t) \leq \sum_{j=1}^m \theta_{i,j}(s_r^t)\, \pi_j^*(s_r^t)\}$, where $\pi_j^*(s_r^t)$ is the stream of profits of the j-th firm determined at date-event s_r^t. The choice of the j-th firm at date-event s_r^t is represented by the supply function $s_j(p(s_r^t))$ that solves the problem $\max\limits_{y_j(s_r^t)} \sum_{s_r^b \in CS(s_r^t)} \pi_j(s_r^t)\, p_{1,s_r^b,s_r^t}(s_r^t)$ subject to the constraint $\sum_{s_r^b \in CS(s_r^t)} (y_{j,h,s_r^b,s_r^c}(s_r^t))_{h \in H; s_r^c \in CS(s_r^t)} \in Y_j(s_r^t)$, where the stream of profits $\pi_j(s_r^t) = (\sum_{h \in H; s_r^c \in CS(s_r^t)} p_{h,s_r^b,s_r^c}(s_r^t) y_{j,h,s_r^b,s_r^c}(s_r^t))_{s_r^b \in CS(s_r^t)}$ is determined by the choice in s_r^t: the firm maximizes the present value of this stream accounting for the forward price of numeraire. Consequently, $\pi_j^*(s_r^t) = (\sum_{h \in H; s_r^c \in CS(s_r^t)} p_{h,s_r^b,s_r^c}(s_r^t) y_{j,h,s_r^b,s_r^c}{}^*(s_r^t))_{s_r^b \in CS(s_r^t)}$ where $y_j^*(s_r^t) \in s_j(p(s_r^t))$. The choice of the i-the consumer at date-event s_r^t is represented by excess demand function $e_i(p(s_r^t)) = \{z_i(s_r^t) \in B_i(p(s_r^t), (\pi_j^*(s_r^t))_{j=1}^m)$: $x_i(s_r^t) \succsim_i^{s_r^t} x_i'(s_r^t)$, where $x_i(s_r^t) = \omega_i(s_r^t) + \sum_{s_r^b \in CS(s_r^t)} z_i(s_r^t)$ and $x_i'(s_r^t) = \omega_i(s_r^t) + \sum_{s_r^b \in CS(s_r^t)} z_i'(s_r^t)$, for every $z_i'(s_r^t) \in B_i(p(s_r^t), (\pi_j^*(s_r^t))_{j=1}^m)\}$. The sequence of temporal competitive equilibria $((z_i^*(s_r^t))_{i=1}^n, (y_j^*(s_r^t))_{j=1}^m, p^*(s_r^t))_{s_r^t \in CS}$ is defined for every $s_r^t \in CS$ by $z_i^*(s_r^t) \in e_i(p^*(s_r^t))$ for every $i = 1,\ldots, n$, $y_j^* \in s_j(p^*(s_r^t))$ for every $j = 1,\ldots, m$, $\sum_{i=1}^n z_i^*(s_r^t) \leq \sum_{j=1}^m y_j^*(s_r^t)$ and $\sum_{i=1}^n \zeta_{i,j}(s_r^t) = 0$ for every $j = 1,\ldots, m$.

When we consider the temporary equilibrium at date-event s_1^0 and confront it with intertemporal equilibrium (assuming that $X_i(s_1^0) = X_i$, $Y_j(s_1^0) = Y_j$, $\sum_{s_r^c \in CS} \omega_{i,s_r^c} = \omega_i$ and $\sum_{s_r^c \in CS} \theta_{i,j,s_r^c} = \theta_{i,j}$ for every $i = 1,\ldots, n$ and $j = 1,\ldots, m$), we see that in the first one just like in the second one consumptions and productions can be chosen for every date-event. The difference is only in the presence of higher number of exchange possibilities, substantially due to the possibility to execute payments at future date-events (while in the intertemporal economy they have to be done at the initial date) and in the possibility to postpone sales and purchases with reopening of the markets in the following temporary equilibria. This difference can lead to a different sequence of consumptions and productions in the distinguished cases, that is in intertemporal equilibrium and in sequential equilibrium. For example, if in sequential economy expectations of prices do not coincide among different consumers, they can start speculating (who expects a higher price for a good buys it in the future market with the intention to sell it in the future in the spot market with a profit, and vice versa for the one who expects a lower price). As a result, there are wealth transfers among the consumers which in turn lead to change in their following consumptions. In contrast, in the intertemporal economy such speculations are impossible. Moreover, even if all the consumers have the same price expectations, but we allow them to be wrong (for example, everybody knows his preferences but not the preferences for the others so nobody is able to form a correct model), the consumers are forced to revise their consumption plans and the sequence of resulting consumptions is not, in general, equal to the one with correct expectations.

Given what we have established so far, we can deduce that we can prove that consumption and production allocations are the same in the two types of equilibrium only by assuming that in temporary equilibrium agents' expectations are correct and coincide, that is at every date-event there is perfect information about prices that occur at the following date-events. There are, of course, other conditions, among them temporal dynamic consistency of the preferences (introduced in the beginning of Chapter 6 and in Paragraph 7.9). We can, now, introduce the following proposition, that considers intertemporal economy $\mathcal{E} = (\langle X_i, \succsim_i\rangle, Y_j, \omega_i, \theta_{i,j}, i = 1,\ldots, n, j = 1,\ldots, m)$ and sequential economy $\mathcal{E}(s_r^t) = (\langle X_i(s_r^t), \succsim_i^{s_r^t}\rangle, Y_j(s_r^t), \omega_i(s_r^t), \theta_{i,j}(s_r^t), i = 1,\ldots, n, j = 1,\ldots, m)$ with $s_r^t \in CS$.

Proposition 12.21 The same consumption and production allocations are obtained for both equilibria, i.e. the sequence of competitive temporary equilibria (or Radner equilibrium) and the competitive intertemporal equilibrium (or Arrow-Debreu equilibrium), with complete markets if:

a) for every firm, every production feasible given the intertemporal production set is also feasible given the sequence of temporary production sets and vice versa, that is $Y_j = \sum_{s_r^t \in CS} Y_j(s_r^t)$ for every $j = 1,\ldots, m$;

b) for every consumer, $X_i = \sum_{s_r^t \in CS} X_i(s_r^t)$, preferences in sequential economy satisfy temporal dynamic consistency and the systems of preferences $(\langle X_i(s_r^t), \succsim_i^{s_r^t}\rangle)_{s_r^t \in CS}$ and $\langle X_i, \succsim_i\rangle$ stand for the same preference relation for every pair $x_i, x_i' \in X_i$;

c) original (that is, not generated by exchanges) endowments of goods and shares in the sequential economy coincide with the endowments in intertemporal economy. That is $\omega_i = (\omega_{i,h,s_r^c})_{h \in H, s_r^c \in CS}$ and $\theta_{i,j} = (\theta_{i,j,s_r^c})_{s_r^c \in CS}$ for every $i = 1,\ldots, n$ and $j = 1,\ldots, m$;

d) all agents' price expectations are correct (or self-fulfilled, or rational).

Proof. We provide intuitive instead of formal proof here. On one hand, since price expectations are correct (and agents know it) we get a sequence of no-arbitrage conditions on the prices that make the equilibrium prices in the intertemporal economy and in sequential economy equivalent. On the other hand, production and consumption sets as well as preferences and endowments are equivalent in those two economies (intertemporal and sequential) and there does not exist a sequence of consumptions or productions allowed in sequential economy that would be excluded in intertemporal economy, and vice versa. Therefore, productions and consumptions chosen by the agents with respect to prices must coincide in these two types of economies. Consequently, competitive equilibria of these two economies are represented by equivalent systems of prices and by the same consumption and production allocations. □

Proposition 12.21 has interesting implications with respect to the number of prices. In sequential, pure exchange economies with correct expectations, in order to reach the sequence of equilibrium productions and consumptions, it is sufficient that in every temporary equilibrium spot prices for all the goods and forward prices for the numeraire good (Arrow-Debreu bonds) only for every date-event immediately after the one that is examined are formed, that is, in every $s_r^t \in CS$, the prices $p_{h,s_r^t,s_r^t}(s_r^t)$ for every h = 2,…, H and $p_{1,s_r^t,s_r^{t+1}}(s_r^t)$ for every $s_r^{t+1} \in CS(s_r^t)$. The number of these prices, that is nevertheless still high, is much lower than the number necessary in the equivalent intertemporal economy (that requires forward prices for all the goods for delivery in every $s_r^t \in CS$). For a production economy the situation is analogous (note that markets for property shares of firms do not necessarily need to exist), with one remark. If productions $y_j(s_r^t)$ include goods (input or output) available also at future date-events other than date–events immediately after s_r^t, then we need to consider also the prices of Arrow-Debreu bonds for all those date-events. If, instead, the list of goods contains also semi-finished products (that subsequently use some inputs and produce outputs), then Arrow-Debreu bonds can be limited to date-events occurring only immediately after the examined one, but the number of goods must be higher.

Notice that the existence of only one type of asset, Arrow-Debreu bonds, is sufficient in order to have complete markets in a sequential economy (so that all agents can take in every period steps to achieve consumptions and productions chosen for this period and for successive date-events). It is therefore interesting to examine a situation with additional and/or substitute assets.

An *asset* is, in general, a good that gives right to receive (and so an obligation to deliver for issuing party) contracted quantities of goods (numeraire, real goods or assets, in which case we talk about *derivatives*, for example options) at future date-events, as established by the asset.

If there are *portfolios* of assets that allow to obtain returns in the same date-events, then no-arbitrage conditions occur for their prices, as shown previously for those particular assets with prices indicated as forward and future prices. Arrow-Debreu bonds introduced before (in every period only one Arrow-Debreu bond for every date-event immediately after), together with spot markets for all the goods and with correct expectations for future date-events prices, are sufficient condition to have complete markets. In general, for the same situation (spot markets for all the goods and correct expectations of future date-events prices), if there are only assets that last one period, it is enough and necessary that the rank of *return matrix* is equal to the number of possible events (in period immediately after the examined period).

For example, with respect to Figure 12.15, let's examine temporary equilibrium at date-event s_2^1, that has immediately afterwards two possible date-events s_2^2 and s_3^2. Then the return matrix has to have rank equal to 2. The return matrix has as many rows as the number of assets and as many columns as the number of events, i.e. 2 columns. Every row indicates the returns (in units of numeraire) of the corresponding asset at any of the date-events s_2^2 and s_3^2. If there are only Arrow-Debreu bonds, that, respectively, expire at date-events s_2^2 and s_3^2, then the indicated matrix is the identity matrix. If there are four assets, of the which the first one has return equal to 1 in both date-events (therefore, it is a *riskless* or *safe* asset), etc., as shown in the following matrix

$$\begin{bmatrix} 1 & 1 \\ 0 & 2 \\ 0 & 4 \\ 0.2 & 0 \end{bmatrix}$$

since its rank is equal to 2, we can be certain that markets are complete. Prices of these assets are not, however, independent. Prices of two assets that define, in the indicated matrix, a non singular minor of order 2 (for example, the first two assets, but not the second and third one) are independent. In equilibrium by no-arbitrage conditions the third asset has price two times higher than the second one (because two units of the second asset give the same return as one unit of the third asset) and that the price of the first asset is five times higher than the price of the fourth asset plus half of the price of the second asset's price (because having 5 units of the fourth asset and a half of the second assets gives the same returns as one unit of the first asset)[17]: so we have the two relationships $p_3 = 2p_2$ and $p_1 = 0.5p_2 + 5p_4$.

[17] If R is a return matrix with a rows and r columns (where $a \geq r$) and there is a non singular minor R_r of order r in R, then we have $p_d = R_d R_r^{-1} p_r$, where p_r is the price vector of the r assets corresponding to the rows of R_r, matrix R_d is the minor corresponding to the other $a - r$ assets, and p_d is their price vector (these prices depend on the prices of the r assets assumed as independent). Note that $R_r^{-1} p_r$ would be the prices of the Arrow-Debreu bonds in terms of p_r, since $p_r = R_r p_{AD}$, where p_{AD} is the price vector of the Arrow-Debreu

Possibility to create derivative assets allows complete markets to emerge also when there are not enough primary assets. Assume that there are, for example, three possible events and one asset with returns [1, 3, 6]. Let's introduce two derivative assets, of the type *call option*. The first one gives right to buy the primary asset (i.e. the asset with returns [1, 3, 6]) at *strike price* 2 and the second one at strike price 5. Then, the first call has returns [0, 1, 4] (that is, in the event that the primary asset has return lower than 2 option is not exercised, if it is higher than 2 the option is exercised by paying the strike price equal to 2, for which the return of the call is equal to the return on the primary asset minus 2). The second call has returns [0, 0, 1]. Thus there are three assets and the return matrix has rank equal to 3, that is the number of possible events. It is not required that there are primary assets that give a positive return in every event. It is enough that returns are different in different events. If, for example, there are two possible events and there is only one Arrow-Debreu bond with returns [1, 0], then introducing one *put* option that allows to sell the primary asset (that is the Arrow-Debreu bond) at price 0.4, we get that this put has returns [0, 0.4]. (That is, if the primary asset has return higher than 0.4, the option is not exercised and if it is lower than 0.4 it is exercised and price 0.4 is collected). Thus, there are two assets with a return matrix with rank equal to 2, that is the number of possible events.

Proposition 12.21 and subsequent observations seem to lead us to conclude that sequential economy can be represented through intertemporal economy. Nevertheless, there are some aspects that make such conclusion rather problematic. In fact assumptions from the Proposition 12.21 are rather strong, in particular the one that requires correct expectations. This assumption implies also that there are no novelties as time passes by (new technologies, new goods, changes in preferences and in number of consumers, unforeseen destruction of resources, etc.), that would determine a sequence of temporary equilibria with consumption and production allocations and prices that cannot always be represented with an intertemporal equilibrium.

In what follows we consider two analyses. The first one deals with a sequential economy where births and deaths of consumers are taken into account: that is we will consider the overlapping generations models. Equilibrium of this sequential economy can be, with appropriate assumptions, described as an intertemporal equilibrium. In the second one, the markets are not complete, that is there does not exist a sufficient number of goods (assets) in order to allow for welfare transfers for every possible future date-event. This situation, in fact, corresponds to reality, if we

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bonds. With respect to the above example, taking $R_r = \begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix}$ and $R_d = \begin{bmatrix} 0 & 4 \\ 0.2 & 0 \end{bmatrix}$, we get $\begin{bmatrix} p_3 \\ p_4 \end{bmatrix} = \begin{bmatrix} 0 & 4 \\ 0.2 & 0 \end{bmatrix} \begin{bmatrix} 1 & -0.5 \\ 0 & 0.5 \end{bmatrix} \begin{bmatrix} p_1 \\ p_2 \end{bmatrix}$.

account for the number of the possible future states of the nature in real world.

12.8 Overlapping generations models

Overlapping generations models describe economies with agents that live for a given period of time. Therefore, they introduce deaths and births of the agents and allow agents of different age to coexist in every period. Assuming that the length of life is known (and, for simplicity, equal for everybody) the set of agents that are born at a certain time (and die at some point of time in the future) is called a generation. We talk about *overlapping generations* because in the same period of time there exist agents belonging to different generations. In this way, we describe an infinitely lasting economy (therefore with infinite number of goods and agents), but composed of a finite number of agents (of different generations) and goods in every period.

In Figure 12.16 we represent agents in a simple economy with overlapping generations. Every generation lasts for two periods of time and in every period there are agents belonging to two different generations: young agents (those born in the current period) and old agents (born in the preceding period).

Time periods

Generations	1	2	3	4	5	6
0	Old					
1	Young	Old				
2		Young	Old			
3			Young	Old		
4				Young	Old	
5					Young	Old
6						Young

Figure 12.16

The infinite number of agents and goods (not in every period but overall) has important consequences for competitive equilibrium. Namely, it may be impossible to determine competitive equilibrium because there is an infinite number of equilibria and each of them is not locally unique. Moreover, equilibrium, determinable or not, may be inefficient. Even only for the sole existence of competitive equilibrium we would need to impose additional (though not particularly strong) assumptions.

We will now try to understand why this problem arises and the impact of infinite number of goods and agents on it. In order to do it, we study four examples of pure exchange economy with constant number of agents (stationary population) that live for two periods of time (as shown in Figure 12.16). These assumptions are maintained (if not specified otherwise) in the remaining of the paragraph.

First, let's extend Definitions 11.1-11.4 to economies with overlapping generations.

Definition 12.9 A private ownership, pure exchange economy with overlapping generations (with agents that live for two periods) is represented by $\mathcal{E} = (\langle X^{i,\tau}, \succsim_{i,\tau}\rangle, \omega^{i,\tau}, i = 1,\ldots, n_\tau, \tau = 0, 1,\ldots, \infty)$, where $\langle X^{i,\tau}, \succsim_{i,\tau}\rangle$ is the system of preferences of the i-th agent from generation τ over his consumption set $X^{i,\tau} \subset \mathbb{R}^{2k}$ (where k denotes the number of goods), $\omega^{i,\tau} = (\omega^{i,\tau}_{h,t})_{h=1}^{k}{}_{t=\tau}^{\infty}$ is the endowment of the same agent and n_τ is the number of agents in generation τ.

Let's refer all prices to the same date, period 1, assuming that there is no difference between intertemporal and sequential equilibrium (in terms of Proposition 12.21). Then, price $p_{h,t}$ denotes price of the h-th good available at time t and paid for in period 1. These are not the prices used in exchanges, because overlapping generations models are intrinsically sequential models, since future generations are not present in period 1. Given that the agents live for two periods, prices used by agents from generation t are $p_{h,t}(t)$ and $p_{h,t+1}(t)$, that are prices paid at period t, for goods $h = 1,\ldots, k$ available, respectively, in period t and t+1 (that is, how many accounting units have to be paid in period t to get in period t a unit of good (h, t) and to obtain in period t+1 a unit of good (h, t+1)). In intertemporal equilibrium we imagine that agents from all the generations are present in initial period 1 and in this period they decide on all the exchanges, carrying out the corresponding payments and receipts. Perfect foresight of prices and no-arbitrage condition imply $\dfrac{p_{h,t+1}}{p_{h,t}} = \dfrac{p_{h,t+1}(t)}{p_{h,t}(t)}$. Budget set of consumer (i, τ), in relation to price matrix $p = (p_{h,t})_{h=1}^{k}{}_{t=1}^{\infty} > 0$, in presence of free disposal, is

$$B^{i,\tau}(p) = \{x^{i,\tau} \in X^{i,\tau} : \textstyle\sum_{h=1}^{k} p_{h,\tau} x^{i,\tau}_{h,\tau} + \sum_{h=1}^{k} p_{h,\tau+1} x^{i,\tau}_{h,\tau+1} \le \sum_{t=\tau}^{\infty} \sum_{h=1}^{k} p_{h,t}\omega^{i,\tau}_{h,t}\}$$

for every $\tau > 0$ and

$$B^{i,0}(p) = \{x^{i,0} \in X^{i,0} : \textstyle\sum_{h=1}^{k} p_{h,1} x^{i,0}_{h,1} \le \sum_{t=1}^{\infty} \sum_{h=1}^{k} p_{h,t}\omega^{i,0}_{h,t}\} \qquad \text{for } \tau = 0.$$

The choice of agent (i, τ) is represented by demand set

$$d^{i,\tau}(p) = \{x^{i,\tau} \in B^{i,\tau}(p) : x^{i,\tau} \succsim_{i,\tau} \tilde{x}^{i,\tau} \text{ per ogni } \tilde{x}^{i,\tau} \in B^{i,\tau}(p)\}.$$

Definition 12.10 A competitive equilibrium is represented by a price matrix $p^* = (p_{h,t}{}^*)_{h=1}^{k}{}_{t=1}^{\infty}$ and an allocation $x^* = (x^{i,\tau}{}^*)_{i=1}^{n_\tau}{}_{\tau=0}^{\infty}$ with $x^{i,\tau}{}^* = (x_{h,t}^{i,\tau}{}^*)_{h=1}^{k}{}_{t=\tau}^{\tau+1}$, such that $x^{i,\tau}{}^* \in d^{i,\tau}(p^*)$ for every $i = 1,\dots, n_\tau$ and τ = 0, 1, …, ∞, and $\sum_{i=1}^{n_t} x_{h,t}^{i,t}{}^* + \sum_{i=1}^{n_{t-1}} x_{h,t}^{i,t-1}{}^* \le \Omega_{h,t}$, where $\Omega_{h,t} = \sum_{\tau=0}^{t}\sum_{i=1}^{n_\tau}\omega_{h,t}^{i,\tau}$ for every $h = 1,\dots, k$ and $t = 1,\dots, \infty$.

We will now examine four examples. In the first example, there is a unique competitive equilibrium, that is moreover efficient. In the other three examples the following problems arise: a competitive equilibrium does not exist in second example; there are multiple equilibria (equilibrium cannot be determined) in the third example; equilibrium is unique but not efficient in the fourth example.

Example with one competitive equilibrium that is efficient. All the generations are composed of an equal (finite) number of agents. All of the agents in every generation are of the same type and there are two perishable consumption goods in every period of time. We set the number of agents in every generation equal to 1 (that is all the quantities of goods considered are, from now on, per agent of a generation). The agents in the generations after 0 generation, that is agents from generations $\tau \ge 1$, have preferences represented by utility functions $u^\tau = x_{1,\tau}^\tau\, x_{1,\tau+1}^\tau\, x_{2,\tau}^\tau\, x_{2,\tau+1}^\tau$ (where $x_{h,t}^\tau$ denotes the quantity of the h-th good available at time t consumed by the agent of generation τ) and endowments $\omega^\tau = (\omega_t^\tau)_{t=\tau}^{\infty}$, with $\omega_\tau^\tau = (2,0)$, $\omega_{\tau+1}^\tau = (0,2)$ and $\omega_t^\tau = (0,0)$ for every $t \ge \tau+2$. The agents from 0 generation have utility functions $u^0 = x_{1,1}^0\, x_{2,1}^0$ and endowment $\omega^0 = (\omega_t^0)_{t=1}^{\infty}$, with $\omega_1^0 = (0,2)$ and $\omega_t^0 = (0,0)$ for every $t \ge 2$. Then, resources are equal to $\Omega_t = (2,2)$ in every period of time, that is for $t \ge 1$. The budget constraint of agents from generation t is $p_{1,t}x_{1,t}^t + p_{1,t+1}x_{1,t+1}^t + p_{2,t}x_{2,t}^t + p_{2,t+1}x_{2,t+1}^t \le 2p_{1,t} + 2p_{2,t+1}$. The budget constraint of agents from 0 generation is $p_{1,1}x_{1,1}^0 + p_{2,1}x_{2,1}^0 \le 2p_{2,1}$. Consequently, the agents from generations $t \ge 1$ choose $x_{1,t}^t = \frac{1}{2}\frac{p_{1,t}+p_{2,t+1}}{p_{1,t}}$, $x_{2,t}^t = \frac{1}{2}\frac{p_{1,t}+p_{2,t+1}}{p_{2,t}}$, $x_{1,t+1}^t = \frac{1}{2}\frac{p_{1,t}+p_{2,t+1}}{p_{1,t+1}}$, $x_{2,t+1}^t = \frac{1}{2}\frac{p_{1,t}+p_{2,t+1}}{p_{2,t+1}}$; while the agent from generation 0 chooses $x_{1,1}^0 = \frac{p_{2,1}}{p_{1,1}}$, $x_{2,1}^0 = 1$. Equilibrium in periods $t > 1$ requires $x_{1,t}^t + x_{1,t}^{t-1} = 2$, that is $\frac{1}{2}\frac{p_{1,t}+p_{2,t+1}}{p_{1,t}} + \frac{1}{2}\frac{p_{1,t-1}+p_{2,t}}{p_{1,t}} = 2$, for the first good, and, for the second good, $x_{2,t}^t + x_{2,t}^{t-1} = 2$, that is $\frac{1}{2}\frac{p_{1,t}+p_{2,t+1}}{p_{2,t}} + \frac{1}{2}\frac{p_{1,t-1}+p_{2,t}}{p_{2,t}} = 2$, so $p_{1,t-1} + p_{1,t} + p_{2,t} + p_{2,t+1} = 4p_{1,t} = 4p_{2,t}$.

Then, we get that $p_{1,t} = p_{2,t}$ for every $t > 1$ and $p_{h,t+1} - 2p_{h,t} + p_{h,t-1} = 0$ for $h = 1, 2$ and for every $t > 1$. This equation has general solution $p_{h,t} = \beta_h\, t + \gamma_h$ for $t \geq 1$, where β_h and γ_h are arbitrary constants, to be determined by initial conditions. In period 1 equilibrium requires $x^0_{1,1} + x^1_{1,1} = 2$ and $x^0_{2,1} + x^1_{2,1} = 2$, that is $\frac{p_{2,1}}{p_{1,1}} + \frac{1}{2}\frac{p_{1,1} + p_{2,2}}{p_{1,1}} = 2$ and $1 + \frac{1}{2}\frac{p_{1,1} + p_{2,2}}{p_{2,1}} = 2$, that is $p_{1,1} = p_{2,1} = p_{2,2}$. Then, equation $p_{1,t} = p_{2,t}$ for every t and the above indicated solution of the difference equation require $\beta_1 = \beta_2 = 0$ and $\gamma_1 = \gamma_2 = \gamma$, that is $p_{1,t} = p_{2,t} = p_{1,1}$ for every $t \geq 1$. Therefore, the competitive equilibrium is represented by price matrix $p^* = (p_{h,t}{}^*)^{2}_{h=1}\,{}^{\infty}_{t=1}$, where $p_{h,t}{}^* = \gamma$, and allocation $x^* = (x^{\tau *})^{n_\tau}_{i=1}\,{}^{\infty}_{\tau=0}$ with $x^{i,\tau}{}^* = (x^{\tau}_{h,t}{}^*)^{2}_{h=1}\,{}^{\tau+1}_{t=\tau}$, where $x^{\tau}_{h,t}{}^* = 1$. This allocation is also efficient. In fact, in order to improve one agent's situation, leaving others' situation unchanged, we would need to give him more than determined by competitive equilibrium of one good, taking away from a young agent from the next generation. This agent would have to be compensated when he is old by an additional quantity of this good taken away from a young agent from the next generation, and so on. This occurs to be impossible. Let's imagine to increase the utility of one agent from generation τ increasing by ε_τ the quantity $x^{\tau}_{1,\tau+1}{}^* = 1$, to obtain $\hat{x}^{\tau}_{1,\tau+1} = 1 + \varepsilon_\tau$. Then, the agent from the following generation has $\hat{x}^{\tau+1}_{1,\tau+1} = 1 - \varepsilon_\tau$ and to be compensated, remembering $u^\tau = x^\tau_{1,\tau}\, x^\tau_{1,\tau+1}\, x^\tau_{2,\tau}\, x^\tau_{2,\tau+1}$, he would need $\hat{x}^{\tau+1}_{1,\tau+2} = 1 + \varepsilon_{\tau+1}$, with $(1-\varepsilon_\tau)(1+\varepsilon_{\tau+1}) = 1$, that is $\varepsilon_{\tau+1} = \frac{1}{1-\varepsilon_\tau} - 1$, and so on. Therefore we have $\varepsilon_{t+1} = \frac{1}{1-\varepsilon_t} - 1$ for every $t \geq \tau$. The solution of this difference equation diverges, if $0 < \varepsilon_\tau < 1$, for $t \to \infty$, as illustrated in Figure 12.17. Then, for sufficiently high t, we get that $\hat{x}^{t+1}_{1,t+1} = 1 - \varepsilon_t \leq 0$ and this proves that such a compensation is impossible.

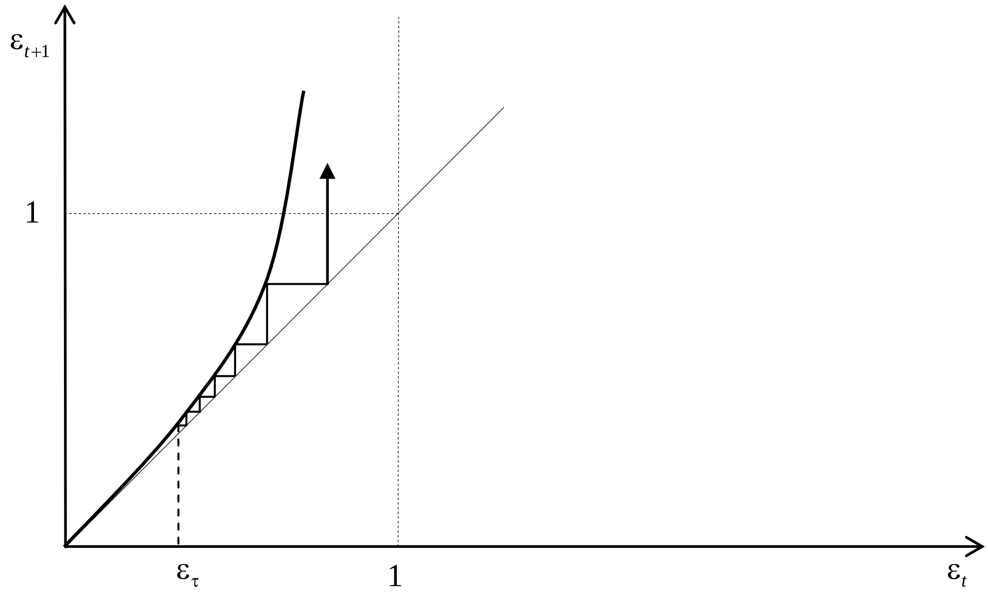

Figure 12.17

Example without competitive equilibria. All the generations are composed of an equal number of agents (for example equal to one). All agents are of the same type and there is only one perishable good in every period of time. Agents from generations $\tau \geq 1$ have preferences represented by utility function $u^\tau = x_\tau^\tau + 3x_{\tau+1}^\tau$ (where x_τ^τ and $x_{\tau+1}^\tau$ indicate, respectively, quantity of good available at time τ and at time τ+1) and endowment $\omega^\tau = (\omega_t^\tau)_{t=\tau}^\infty$ with $\omega_\tau^\tau = \omega_{\tau+1}^\tau = 1$ and $\omega_t^\tau = 0$ for every $t > \tau+1$. Agents from generation 0 have preferences represented by utility function $u^0 = x_1^0$ and endowment $\omega^0 = (\omega_t^0)_{t=1}^\infty$ with $\omega_t^0 = 2^{-t}$ for every t (endowment with infinite elements is crucial, in this example, for equilibrium inexistence). Then in period 1 resources are equal to $\Omega_1 = \dfrac{3}{2}$ and in periods $t > 1$ they are equal to $\Omega_t = 2 + 2^{-t}$. Indicating with p_t the price of good available at time t, the budget constraint of an agent from generation t (with $t \geq 1$) is $p_t x_t^t + p_{t+1} x_{t+1}^t \leq p_t + p_{t+1}$, while the one of agents from generation 0 is $p_1 x_1^0 \leq \sum_{t=1}^\infty 2^{-t} p_t$.[18] In order to prove that competitive equilibrium does not

[18] It should not be surprising that an agent from generation 0 manages to sell a good available at time $t > 2$ (so when he is dead) to an agent from generation 1 (also dead in period $t > 2$). These goods are sold by agents from generation 0 to agents from generation 1, then from them to those from generation 2, and so on, until they become a property of agents from generation t (that consume it). This happens in real world too. Consider a market of irredeemable bonds (i.e. not terminable by payment of the principal, but giving perpetual returns), through which agents exchange flows of returns the most part of which mature after their death.

exist imagine, on the contrary, that it exists. Since, for every good there is at least one agent with preferences that are strongly monotone with respect to this good, all prices have to be positive in equilibrium. The choice of agents from generation t is $x_t^t = 0$, $x_{t+1}^t = 1 + \frac{p_t}{p_{t+1}}$ if $\frac{p_{t+1}}{p_t} < 3$; $x_t^t = 1 + \frac{p_{t+1}}{p_t}$, $x_{t+1}^t = 0$ if $\frac{p_{t+1}}{p_t} > 3$; undetermined, with $x_t^t \geq 0$, $x_{t+1}^t \geq 0$ and $x_t^t + 3x_{t+1}^t = 4$, if $\frac{p_{t+1}}{p_t} =$ 3. We prove that $\frac{p_{t+1}}{p_t} < 3$ for every $t \geq 1$. In order to do that we introduce the set T' of periods of time $t > 1$ for which we have $\frac{p_{t+1}}{p_t} < 3$, that is $T' = \{t > 1 : \frac{p_{t+1}}{p_t} < 3\}$. First, we find out that this set is not finite. In fact, if it is finite, there would be a $\hat{t}$ with $\frac{p_{t+1}}{p_t} \geq 3$ for every $t \geq \hat{t}$. Then, the endowment of an agent from generation 0 would have an infinite value with respect to the price of the good in period 1, since $\frac{1}{p_1}\sum_{t=1}^{\infty} 2^{-t} p_t =$ $\frac{1}{p_1}\sum_{t=1}^{\hat{t}-1} 2^{-t} p_t + \frac{1}{p_1}\sum_{t=\hat{t}}^{\infty} 2^{-t} p_t$ with $\frac{1}{p_1}\sum_{t=\hat{t}}^{\infty} 2^{-t} p_t = 2^{-\hat{t}}\frac{p_{\hat{t}}}{p_1}\left(1 + \sum_{s=1}^{\infty} 2^{-s}\frac{p_{\hat{t}+s}}{p_{\hat{t}}}\right) =$ $= 2^{-\hat{t}}\frac{p_{\hat{t}}}{p_1}\left(1 + \sum_{s=1}^{\infty} 2^{-s}\prod_{r=1}^{s}\frac{p_{\hat{t}+r}}{p_{\hat{t}+r-1}}\right) \geq 2^{-\hat{t}}\frac{p_{\hat{t}}}{p_1}\left(1 + \sum_{s=1}^{\infty}\left(\frac{3}{2}\right)^s\right) = \infty$. As a result, this agent would demand an infinite quantity of good 1, that is $x_1^0 = \infty$, that is incompatible with availability $\Omega_1 = \frac{3}{2}$. Also, there cannot exist a $t \geq 2$, with $\frac{p_t}{p_{t-1}} \geq 3$ and $\frac{p_{t+1}}{p_t} < 3$, because in such a case demand for good t, expressed by individuals from generations t–1 and t, would be, by individuals from generation t–1, equal to $x_t^{t-1} = 0$ if $\frac{p_t}{p_{t-1}} > 3$ or to $x_t^{t-1} \in [0, \frac{4}{3}]$ if $\frac{p_t}{p_{t-1}} = 3$, and, by individuals from generation t, equal to $x_t^t = 0$. Thus a quantity lower than the available quantity $\Omega_t = 2 + 2^{-t}$ of the good t would be demanded, in contrast with equilibrium condition that requires equality between demand and supply (recalling that the preferences are strongly monotone). Therefore, it must be that $\frac{p_{t+1}}{p_t} < 3$ for every $t \geq 1$. Let's examine the market for good t: demand is $x_t^{t-1} > 0$ and $x_t^t = 0$, so the equilibrium condition $x_t^{t-1} + x_t^t = \Omega_t = 2 + 2^{-t}$ implies that $x_t^{t-1} > 2$. Then, considering the budget constraint of generation t (with $t \geq 1$), that requires $p_t x_t^t + p_{t+1} x_{t+1}^t \leq p_t + p_{t+1}$, and having $x_t^t = 0$, we get the inequalities $p_t + p_{t+1} \geq p_{t+1} x_{t+1}^t > 2 p_{t+1}$, so with $p_{t+1} < p_t$ for every $t \geq 1$. Inequality $p_{t+1} < p_t$ for every $t \geq 1$, however, implies that an agent from generation 0 demands a quantity of good 1 equal to $x_1^0 = \frac{1}{p_1} \sum_{t=1}^{\infty} 2^{-t} p_t = \frac{1}{2} + \sum_{t=2}^{\infty} 2^{-t} \prod_{s=1}^{t} \frac{p_{s+1}}{p_s} < \frac{1}{2} + \sum_{t=2}^{\infty} 2^{-t} = 1$. Consequently, demand for good 1, that is equal to 0 for agents from generation 1, is smaller than the available quantity $\Omega_1 = \frac{3}{2}$. Therefore, a competitive equilibrium does not exist for this economy.

Example without a determined competitive equilibrium. All the generations $t \geq 1$ are composed of an equal number of two types of agent. (for example one agent of each type). Generation 0 is composed of only one type of agent (in order to keep the population stationary, there must be two agents of this type). In every period of time there are two perishable consumption goods. Agents from generations $\tau \geq 1$ have preferences represented, respectively for the two types, by utility functions $u^{1,\tau} = x_{1,\tau}^{1,\tau} + 2(x_{1,\tau+1}^{1,\tau})^{1/2}$ and $u^{2,\tau} = 2(x_{2,\tau}^{2,\tau})^{1/2} + x_{2,\tau+1}^{2,\tau}$ (where $x_{h,t}^{i,\tau}$ denotes the quantity of the h-th good consumed at time t by agent of type i from generation τ) and by endowments $\omega^{1,\tau} = (\omega_t^{1,\tau})_{t=\tau}^{\infty}$, with $\omega_\tau^{1,\tau} = (1,0)$ and $\omega_t^{1,\tau} = (0,0)$ for every $t > \tau$, and $\omega^{2,\tau} = (\omega_t^{2,\tau})_{t=\tau}^{\infty}$, with $\omega_{\tau+1}^{2,\tau} = (0,1)$ and $\omega_t^{2,\tau} = (0,0)$ for $t = \tau$ and for every $t > \tau+1$. Every agent from generation 0 has utility function $u^0 = 2^{1/2}(x_{1,1}^0)^{1/2} + x_{2,1}^0$ and endowment $\omega^0 = (\omega_t^0)_{t=1}^{\infty}$, with $\omega_1^0 = (0, \frac{1}{2})$ and $\omega_t^0 = (0,0)$ for every $t \geq 2$. The available quantity of two goods is represented by $\Omega_t = (1,1)$ in every period of time, that is for $t \geq 1$. The budget constraint of agents (1, t) (type 1 from generation t), who are interested only in the first good and consequently make exchanges only of this good. is given by $p_{1,t} x_{1,t}^{1,t} + p_{1,t+1} x_{1,t+1}^{1,t} \leq p_{1,t}$, where $p_{1,t}$ and $p_{1,t+1}$ denote price of good 1 available in, respectively, period t and t+1. The budget constraint of agents (2, t), that exchange only the second good, is $p_{2,t} x_{2,t}^{2,t} + p_{2,t+1} x_{2,t+1}^{2,t} \leq p_{2,t+1}$. The budget constraint of agents from generation 0 is $p_{1,1} x_{1,1}^0 + p_{2,1} x_{2,1}^0 \leq \frac{1}{2} p_{2,1}$. Agent (1, t) chooses $x_{1,t}^{1,t} = 1 - \frac{p_{1,t}}{p_{1,t+1}}$, $x_{1,t+1}^{1,t} = \left(\frac{p_{1,t}}{p_{1,t+1}} \right)^2$; agent (2, t) chooses $x_{2,t}^{2,t} = \left(\frac{p_{2,t+1}}{p_{2,t}} \right)^2$, $x_{2,t+1}^{2,t} = 1 - \frac{p_{2,t+1}}{p_{2,t}}$;

while agent 0 chooses $x_{1,1}^{0}=\frac{1}{2}\left(\frac{p_{2,1}}{p_{1,1}}\right)^{2}$, $x_{2,1}^{0}=\frac{1}{2}-\frac{1}{2}\frac{p_{2,1}}{p_{1,1}}$. Equilibrium in periods $t > 1$ requires, for good 1, $x_{1,t}^{1,t}+x_{1,t}^{1,t-1}=1$, that is $1-\frac{p_{1,t}}{p_{1,t+1}}+\left(\frac{p_{1,t-1}}{p_{1,t}}\right)^{2}=1$, that is $\frac{p_{1,t+1}}{p_{1,t}}=\left(\frac{p_{1,t}}{p_{1,t-1}}\right)^{2}$, so $\frac{p_{1,t+1}}{p_{1,t}}=\left(\frac{p_{1,2}}{p_{1,1}}\right)^{2^{t-1}}$. For the second good, $x_{2,t}^{2,t}+x_{2,t}^{2,t-1}=1$ for every $t > 1$, that is $\left(\frac{p_{2,t+1}}{p_{2,t}}\right)^{2}+$ $1-\frac{p_{2,t}}{p_{2,t-1}}=1$, so $\frac{p_{2,t+1}}{p_{2,t}}=\left(\frac{p_{2,t}}{p_{2,t-1}}\right)^{1/2}$, that is $\frac{p_{2,t+1}}{p_{2,t}}=\left(\frac{p_{2,2}}{p_{2,1}}\right)^{(1/2)^{t-1}}$. Equilibrium in period $t = 1$ requires, for the first good, $2x_{1,1}^{0}+x_{1,1}^{1,1}=1$, that is $\left(\frac{p_{2,1}}{p_{1,1}}\right)^{2}+1-\frac{p_{1,1}}{p_{1,2}}=1$, so $\frac{p_{1,2}}{p_{1,1}}=\left(\frac{p_{2,1}}{p_{1,1}}\right)^{-2}$, and, for the second good, $2x_{2,1}^{0}+x_{2,1}^{2,1}=1$, that is $1-\frac{p_{2,1}}{p_{1,1}}+\left(\frac{p_{2,2}}{p_{2,1}}\right)^{2}=1$, so $\frac{p_{2,2}}{p_{2,1}}=\left(\frac{p_{2,1}}{p_{1,1}}\right)^{1/2}$. Then, since there are two equations for three unknowns, we get that the exchange ratio $\frac{p_{2,1}}{p_{1,1}}$ is arbitrary. (It belongs to the interval $[0,1]$ to have internal solutions: in fact, $x_{1,t}^{1,t}=1-\frac{p_{1,t}}{p_{1,t+1}}$, $x_{2,t+1}^{2,t}=1-\frac{p_{2,t+1}}{p_{2,t}}$ and $x_{2,1}^{0}=\frac{1}{2}-\frac{1}{2}\frac{p_{2,1}}{p_{1,1}}$ respectively imply, in order to have internal points, $\frac{p_{12}}{p_{11}}\geq 1$, $\frac{p_{22}}{p_{21}}\leq 1$ and $\frac{p_{21}}{p_{11}}\leq 1$). Setting $\frac{p_{2,1}}{p_{1,1}}=\alpha^{-1}$ (so $\alpha \geq 1$), we finally get other exchange ratios $\frac{p_{2,2}}{p_{2,1}}=\alpha^{-1/2}$, $\frac{p_{1,2}}{p_{1,1}}=\alpha^{2}$, $\frac{p_{1,t+1}}{p_{1,t}}=\alpha^{2^{t}}$ and $\frac{p_{2,t+1}}{p_{2,t}}=\alpha^{-(1/2)^{t}}$. There is, then, an infinite number of competitive equilibria (one for every $\alpha \geq 1$). Therefore, equilibrium is indetermined.

Example with an inefficient competitive equilibrium. All the generations are composed of an equal number of agents (for example equal to one), all of the same type and there is only one consumption good in every period of time. Agents from generations $\tau \geq 1$ have preferences represented by utility function $u^{\tau}=x_{\tau}^{\tau}+x_{\tau+1}^{\tau}$ (where x_{τ}^{τ} and $x_{\tau+1}^{\tau}$ denote, respectively, quantity of good consumed in time τ and $\tau+1$ by agent from generation τ) and an endowment $\omega^{\tau}=(\omega_{t}^{\tau})_{t=\tau}^{\infty}$ with $\omega_{\tau}^{\tau}=\omega_{\tau+1}^{\tau}=1$ and $\omega_{t}^{\tau}=0$ for every $t > \tau+1$. Agents from generation 0 have preferences represented by utility function $u^0 = x_1^0$ and endowment $\omega^0 = (\omega_t^0)_{t=1}^{\infty}$ with $\omega_1^0 = 1$ and $\omega_t^0 = 0$ for every $t > 1$. Allocation represented by endowments and vector of prices $p^* = (p_t^*)_{t=1}^{\infty}$ with $p_t^* = 1$ for every t constitute a competitive equilibrium. At these prices no agent can increase his utility with sales or purchases. However, there exist feasible Pareto-superior allocations, that is with higher utility for all the agents. Let $0 < \varepsilon < \frac{1}{2}$ and consider allocation that assigns to agents from generation 0 quantity $\hat{x}_1^0 = 1 + \varepsilon$ (so $u^0(\hat{x}_1^0) = 1 + \varepsilon > 1 = u^0(\omega_1^0)$), and to agents from generation t quantity $\hat{x}_t^t = 1 - \sum_{i=1}^{t} \varepsilon^i$, $\hat{x}_{t+1}^t = 1 + \sum_{i=1}^{t+1} \varepsilon^i$ (so $u^t(\hat{x}_t^t, \hat{x}_{t+1}^t) = 2 + \varepsilon^{t+1} > 2 = u^t(\omega_t^t, \omega_{t+1}^t)$). This allocation is better for all the agents and feasible (because $\hat{x}_t^t + \hat{x}_t^{t-1} = 2$ for every t). Therefore, the indicated competitive equilibrium is inefficient.

The last three examples introduce the main terms for the discussion of the overlapping generations analysis.

The existence of a competitive equilibrium is guaranteed by the usual assumption for the finite case and by some new assumptions regarding the case with infinite agents and goods. These assumptions require that the number of agents and goods be countable, aggregate endowment of every good be positive and bounded, every good be desired by a finite number of agents, and consumption and endowment of every agent be referred to a finite number of time periods.[19]

Impossibility to determine equilibrium is probably the most notable feature of overlapping generations models. While in the finite case indeterminacy is not normally robust (as shown in Paragraph 12.1), in overlapping generation models it can be robust. An intuitive explanation is connected with equilibrium feasibility conditions (according to which aggregate demand of every good is equal to available quantity), that are required for every time period t, but not necessarily for $t = \infty$. There can be as many feasibility conditions missing as many goods are present in a time period minus one (being a condition provided by Walras law) and this number is a measure of indeterminacy. (In the example without a determined equilibrium, there are 2 goods in every time period and there is 1 degree of indeterminacy, signaled by the arbitrary value of α. The missing market for $t = \infty$ can be found noting that $\lim_{t \to \infty} p_{1,t}(t) = 0$ and so the budget constraint of agents of type 1 allows them to demand an infinite quantity of the first good at young age at $t = \infty$, while the available quantity is equal to 1). Indeterminacy of equilibrium in these models provides a nice microeconomic foundation to represent some aspects of Keynesian macroeconomy. For example in equilibrium employment can take any value in a certain interval, so involuntary unemployment occurs.

[19] For a more accurate and complete representation see Geanakoplos and Polemarchakis (1991).

Inefficiency of competitive equilibrium illustrated by the fourth example means that the first welfare theorem does not hold. It is interesting to see why the proof of this theorem does not hold in the examined example. Let's take the proof of Proposition 11.6 and apply to the example in which on one hand we have a competitive equilibrium allocation (coinciding with the endowment allocation) and on the other hand a feasible allocation that is Pareto-superior. The first one is $\omega^{\tau} = (\omega_t^{\tau})_{t=\tau}^{\infty}$ with $\omega_1^0 = 1$ and $\omega_t^0 = 0$ for every $t > 1$, and $\omega_{\tau}^{\tau} = \omega_{\tau+1}^{\tau} = 1$ and $\omega_t^{\tau} = 0$ for every $t > \tau+1$. The second one is $\hat{x}^{\tau} = (\hat{x}_t^{\tau})_{t=\tau}^{\infty}$ with $\hat{x}_1^0 = 1+\varepsilon$ and $\hat{x}_t^0 = 0$ for every $t > 1$, and $\hat{x}_{\tau}^{\tau} = 1 - \sum_{i=1}^{\tau} \varepsilon^i$, $\hat{x}_{\tau+1}^{\tau} = 1 + \sum_{i=1}^{\tau+1} \varepsilon^i$ and $\hat{x}_t^{\tau} = 0$ for every $t > \tau+1$. This allocation is feasible because $\hat{x}_t^t + \hat{x}_t^{t-1} = 2$ for every good t. Following the proof from Proposition 11.6 (the first welfare theorem) we induce from this equality (multiplying by p_t and summing up) that $\sum_{t=1}^{\infty} p_t(\hat{x}_t^t + \hat{x}_t^{t-1}) = \sum_{t=1}^{\infty} 2p_t$ for every $p = (p_t)_{t=1}^{\infty}$, that is $\sum_{t=1}^{\infty} p_t \hat{x}_t^t + \sum_{t=0}^{\infty} p_{t+1} \hat{x}_{t+1}^t = \sum_{t=1}^{\infty} p_t + \sum_{t=0}^{\infty} p_{t+1}$, and therefore, there is at least one $t \geq 0$ (that is one agent) for whom $p_t \hat{x}_t^t + p_{t+1} \hat{x}_{t+1}^t \leq p_t + p_{t+1}$ (obviously, $p_1 \hat{x}_1^0 \leq p_1$ for $t = 0$). However, this does not happen in the examined case. In fact, with $p_t{*} = 1$ for every $t = 1, \ldots, \infty$, we find out that $\hat{x}_t^t + \hat{x}_{t+1}^t = 1 - \sum_{i=1}^{t} \varepsilon^i + 1 + \sum_{i=1}^{t+1} \varepsilon^i = 2 + \varepsilon^{t+1} > 2$ for every $t \geq 1$ and $\hat{x}_1^0 = 1 + \varepsilon > 1$. The above implication does not hold because $\sum_{t=1}^{\infty} 2p_t{*}$, that is $\sum_{t=1}^{\infty} p_t{*}\Omega_t$, is infinite in the examined case, while it is bounded in the proof of Proposition 11.6. We can, then, induce that the first welfare theorem (as stated in Proposition 11.6, that considers weak efficiency) holds also for overlapping generations models if equilibrium prices are such that the aggregate value of the endowments, that is $\sum_{t=1}^{\infty} p_t{*}\Omega_t$, is bounded.[20]

Competitive equilibrium allocation is, by Proposition 11.7, strongly efficient if preferences of all the consumers are locally non satiated. In overlapping generations models this property holds if we introduce a bit stronger condition. We need to assume that preferences are monotone (as indicated in Paragraph 3.2, the preferences are monotone if $x' \gg x$ implies $x' \succ x$). The proof can be obtained in an analogous way to the proof of Proposition 11.7.[21] Thus, a competitive equilibrium allocation (for

[20] Note that this condition is sufficient but not necessary. In fact, in the first example in this paragraph competitive equilibrium allocation is efficient even if this condition is not satisfied.

[21] In the proof we need monotonicity instead of local non satiation since, in case of an infinite number of goods, for local non satiation (unlike for monotonicity) the inequality $p * x_i < p * \omega_i$ does not imply existence of a point $x_i' \succ_i x_i$ such that $p * x_i' < p * \omega_i$. Let's consider, for example, the utility function $u_i = \min_{h=1,\ldots,\infty} x_{ih}$, an endowment $\omega_i = (\omega_{ih})_{h=1}^{\infty}$

overlapping generations economy) is strongly efficient if the aggregate value of the endowments (that is $\sum_{t=1}^{\infty} p_t * \Omega_t$, where p* is the equilibrium vector of prices) is bounded and preferences of all the agents are monotone.

The second welfare theorem applies to overlapping generations models without any particular problems. It can be formulated in the following way (that holds also for models with a finite number of generations).

Proposition 12.22 If an allocation $\hat{x} = (\hat{x}^{i,\tau})_{i=1}^{n_\tau}{}_{\tau=0}^{\infty}$, with $\hat{x}^{i,\tau}$ = $(\hat{x}_{h,t}^{i,\tau})_{h=1}^{k}{}_{t=\tau}^{\tau+1}$ (where (i, τ) denotes agent i from generation τ and n_τ the number of agents in this generation) is strongly efficient for a given economy with overlapping generations and this economy allows for competitive equilibrium (x^*, p^*) given the endowments $(\omega_{h,t}^{i,\tau})_{h=1}^{k}{}_{t=\tau}^{\tau+1} = (\hat{x}_{h,t}^{i,\tau})_{h=1}^{k}{}_{t=\tau}^{\tau+1}$, then $(\hat{x}, p^*)$ is also a competitive equilibrium.

Proof. The competitive equilibrium allocation is necessarily such that $x^{i,\tau} * \succsim_{i,\tau} \omega^{i,\tau} = \hat{x}^{i,\tau}$ for every i = 1,…, n_τ and τ = 0, 1,…, ∞. Since allocation $\hat{x} = (\hat{x}^{i,\tau})_{i=1}^{n_\tau}{}_{\tau=0}^{\infty}$ is strongly efficient and allocation x* is feasible, we must have $x^{i,\tau} * \sim_{i,\tau} \hat{x}^{i,\tau}$ for every i = 1,…, n_τ and τ = 0, 1,…, ∞. Therefore, also $(\hat{x}, p^*)$ is a competitive equilibrium. □

with $\omega_{ih} = 2^{-h}$ for h = 1,…, ∞, a bundle of goods $x_i = (x_{ih})_{h=1}^{\infty}$ with $x_{i1} = 2^{-2}$ and $x_{ih} = 2^{-h}$ for h = 2,…, ∞ and a vector of prices $p^* = (1)_{h=1}^{\infty}$. We find out that $p * x_i = \frac{3}{4} < 1 = p * \omega_i$, while $x_i' \succ_i x_i$ requires $\min_{h=1,\dots,\infty} x_{ih}' > 0$, so with $p * x_i' = \infty > 1 = p * \omega_i$.