Aldo Montesano PRINCIPLES OF ECONOMIC ANALYSIS

CHAPTER 11. General equilibrium analysis I

Partial equilibrium analysis, considered in the previous chapter, is an important instrument to study markets for goods in which interdependence can be disregarded as well as a very useful introduction to the analysis of general equilibrium. However, when interdependence is relevant, in other words when the equilibrium values (price and quantity) of the good examined influence the equilibrium values of other goods and vice versa, then partial equilibrium analysis is insufficient. Let's analyze labor market as an example. Wages and employment influence significantly the demand and supply of the consumption goods. Prices and produced quantities of these goods, on the other hand, have large influence on the demand and supply of employment. Therefore, assuming that these influences are not relevant and analyzing the labor market using partial equilibrium analysis would lead to a poor representation of the economic reality. Instead, we need to carry out analysis that will explicitly take into account interdependence and general equilibrium serves this purpose.

General equilibrium analysis examines feasible choices with respect to all goods and all agents. All production levels and exchanges are established and an allocation of all goods to all the agents in the economy is determined. General equilibrium analysis is obviously more complex than partial equilibrium. Not only it requires extensive mathematical reasoning in order to obtain the main results (like equilibrium existence, uniqueness, stability, efficiency and comparative statics), but also the applications of the theory are less specific (for example because of the large number of variables and conditions implied by the simultaneous consideration of all goods and all agents). Moreover, even though studies of non-competitive markets are not lacking in general equilibrium literature, the main part of this literature assumes that markets are competitive and disregards the case of non-competitive markets. This situation depends, on one hand, on the intrinsic difficulty of general equilibrium analysis and, on the other hand, on the consideration that competitive equilibrium allocations serve as an important benchmark for all allocation comparisons.

In the remaining of this and next chapter, we will first define, under some assumptions, general competitive equilibrium (or *Walrasian equilibrium,* that owes its name to Walras, who was the first one to examine it in his *Elements of Pure Economics,* first edition 1874-77). Then we will introduce some analyses regarding, among other aspects, existence, uniqueness and stability of the equilibrium and the core.

11.1 General competitive equilibrium

Competitive equilibrium was introduced in Definition 10.1. The choices of price-taking agents were examined in Chapters 3, 4 and 5. We now need to specify these choices in the general equilibrium framework and ensure their feasibility. Each of the agents, as in Paragraph 4.5, has some endowment of goods, that can be exchanged. The aggregate Walrasian demand function of the consumers is, as shown in Paragraph 4.6, of the type $D(p, p\omega_1,\ldots, p\omega_n)$ and excess demand is given by $D(p, p\omega_1,\ldots, p\omega_n) - \Omega$, with $\Omega = \sum_{i=1}^{n}\omega_i$, where $\omega_i\in\mathbb{R}_+^k$ is the endowment vector of the i-th consumer (k is the number of goods). If we do not have the necessity to evidence the dependence on the endowments, we can denote the aggregate demand function as $D(p)$.

In a pure exchange economy (that is without production) the feasibility condition of the exchange between consumers requires that the overall quantity of the goods that the consumers desire, equal to $D(p)$, is equal to the overall quantity of goods available in the economy, equal to Ω. Therefore, equilibrium condition requires that the prices are such that $D(p^*) = \Omega$. Introducing *aggregate excess demand function* $E(p) = D(p)-\Omega$, this equilibrium condition becomes $E(p^*) = 0$. (If $D(p)$ is a correspondence, the equilibrium condition is $\Omega\in D(p^*)$, that is $0\in E(p^*)$).

In a production economy (in which there are both consumers and producers) another element, aggregate supply function $S(p)$, introduced in Paragraph 5.7, enters. Moreover, the demand function of the consumers takes into account (as will be indicated in Paragraph 11.6) the wealth generated by the profits from production. In equilibrium the following condition must hold $D(p^*) = S(p^*)+\Omega$ (or $\Omega\in D(p^*)-S(p^*)$). Introducing the aggregate excess demand function $E(p) = D(p)-S(p)-\Omega$, the equilibrium condition becomes $E(p^*) = 0$ (or $0\in E(p^*)$).

Equilibrium prices $p^* = (p_1^*,\ldots, p_k^*)$ determine allocation of goods to all consumers and producers through the relationships $x_i^* = d_i(p^*)$ and $y_j^* = s_j(p^*)$, for $i = 1,\ldots, n$ and $j = 1,\ldots, m$, where n is the number of consumers and m is the number of producers. Competitive general equilibrium is represented by a vector of prices p^* and by an allocation $(x^*, y^*) = (x_1^*,\ldots, x_n^*, y_1^*,\ldots, y_m^*)$, that is (p^*, x^*, y^*). According to competitive equilibrium condition, bundles of goods in the allocation need to correspond to choices made by (price-taking) agents at prices p^* and choices need to be feasible.

After this informal introduction, we will move to a formal introduction of general competitive equilibrium. We will distinguish between two cases: with and without free disposal assumption. Free disposal (introduced in Definition 8.3) assumes that each agent can dispose of any quantity of goods without incurring any cost.

11.2 Private ownership economy

In this paragraph we define the *private ownership economy* (the centrally planned economy, which is another type of economy, will be mentioned in Paragraph 11.15). In the following paragraph we examine pure exchange competitive equilibria of a private ownership economy. Production equilibrium will be presented in Paragraph 11.8.

Definition 11.1 (*Private ownership economy*) The economy is composed of consumers represented by their consumption sets and systems of preferences, producers (firms) represented by their production sets and resources (or available goods). In the economy with private ownership consumers own resources and firms.

Therefore, a pure exchange economy with private ownership is represented by $\mathcal{E} = (\langle X_i, \succsim_i \rangle, \omega_i, i = 1,\ldots, n)$, where $\langle X_i, \succsim_i \rangle$ is the system of preferences of the i-th consumer over his consumption set $X_i \subset \mathbb{R}^k$ (where k represents the number of goods), $\omega_i \in \mathbb{R}^k_+$ are the resources in the endowment of the same consumer and n is the number of consumers.

A production economy with private ownership is represented by $\mathcal{E} = (\langle X_i, \succsim_i \rangle, Y_j, \omega_i, \theta_{ij}, i = 1,\ldots, n, j = 1,\ldots, m)$, where, beyond already specified symbols, $Y_i \subset \mathbb{R}^k$ is the production set of j-th firm, θ_{ij} is i-th consumer's share in firm j and m is the number of firms, so $\theta_{ij} \geq 0$ for every $i = 1,\ldots, n$ and $j = 1,\ldots, m$ and $\sum_{i=1}^{n} \theta_{ij} = 1$ for every $j = 1,\ldots, m$.

We note that the pure exchange economy is a particular type of the production economy. Production economy becomes pure exchange if $Y_i = \{0\}$ for every $j = 1,\ldots, m$.

11.3 Pure exchange competitive equilibrium

The pure exchange competitive equilibrium is defined by the feasibility of the consumer's choices in the economy $\mathcal{E} = (\langle X_i, \succsim_i \rangle, \omega_i, i = 1,\ldots, n)$. It is therefore, necessary to define consumers' choices and impose feasibility conditions. Each consumer is constrained to choose a bundle of goods that satisfies his budget set, defined by the prices, his endowment and by the possibility of free disposal, if it is the case.

Definition 11.2 (*Consumer's budget set*) Budget set (that indicates bundles of goods that can be achieved through exchange) of the *i-th* consumer, without free disposal, is

$$\bar{B}_i(p) = \{x_i \in X_i : px_i = p\omega_i\}$$

with respect to a price vector $p\in\mathbb{R}^k$ (that includes the possibility that prices are negative)[1], and in the presence of *free disposal* is

$$B_i(p)=\{x_i \in X_i : x_i \le x_i' \text{ for some } x_i' \in \mathbb{R}^k \text{ such that } px_i' = p\omega_i\}$$

Proposition 11.1 If the prices are non negative, that is $p \ge 0$, then the consumer's budget set in the presence of free disposal (introduced in Definition 11.2) can be represented as

$$B_i(p) = \{x_i \in X_i : px_i \le p\omega_i\}$$

Proof. Consider the budget set, introduced in the Definition 11.2, $B_i(p)=\{x_i \in X_i : x_i \le x_i'$ for some $x_i' \in \mathbb{R}^k$ such that $px_i' = p\omega_i\}$. With respect to this set, on one hand, if $p \ge 0$ and $x_i \in B_i(p)$, then $px_i \le px_i' = p\omega_i$, and, on the other hand, if $p \ge 0$, $x_i \in X_i$ and $px_i \le p\omega_i$, then $x_i \in B_i(p)$ because there exits a $x_i' \ge x_i$ in $\mathbb{R}^k$ such that $px_i' = p\omega_i$, Therefore, if $p \ge 0$, then two specifications of $B_i(p)$ (introduced in Definition 11.2 and Proposition 11.1) coincide. □

Definition 11.3 (*Consumer's choice*) The choice of the consumer represented by the regular system of preferences $\langle X_i, \succsim_i\rangle$, where $X_i \subset \mathbb{R}^k$, and by the endowment $\omega_i\in\mathbb{R}^k_+$, in the absence of free disposal, is a point that belongs to the set

$$\bar{d}_i(p) = \{x_i \in \bar{B}_i(p) : x_i \succsim_i x_i' \text{ for every } x_i' \in \bar{B}_i(p)\}$$

and in the presence of free disposal is a point belonging to the set

$$d_i(p) = \{x_i \in B_i(p) : x_i \succsim_i x_i' \text{ for every } x_i' \in B_i(p)\}$$

Definition 11.4 (*Competitive equilibrium*) Competitive equilibrium requires that choices are feasible. Therefore, in the absence of free disposal, equilibrium is represented by a vector of prices $p^*\in\mathbb{R}^k$ and an allocation $x^* = (x_1^*,\ldots, x_n^*)$ such that $x_i^*\in \bar{d}_i(p^*)$ for every $i = 1,\ldots, n$ and $\sum_{i=1}^n x_i^* = \sum_{i=1}^n \omega_i$. In presence of free disposal, such that $x_i^*\in d_i(p^*)$ for every $i = 1,\ldots, n$ and $\sum_{i=1}^n x_i^* \le \sum_{i=1}^n \omega_i$.

Proposition 11.2 If a pure exchange economy satisfies the free disposal assumption and the consumers are altogether sufficiently greedy, then the equilibrium price vector is semipositive or seminegative, i.e. $p^* > 0$ or $p^* < 0$. The "altogether sufficiently greedy" condition requires that global

[1] The one who buys the good at a negative price collects the money, and the one that sells it pays. Proposition 11.1 excludes goods with negative prices by assuming free disposal. The problem of negative prices arises for goods, sometimes called "bads", that the agent would get rid of than buy. An example of "bad" is garbage. Instead of introducing such a "bad" we can introduce the service for its disposal that has positive price (equal, in the absolute value, to the price of the "bad"). This is what happens in reality. It is not possible, in general, to know whether a thing is "good" or "bad" before the equilibrium is determined. Moreover, if the economy allows for more than one equilibrium, it can happen that the same thing is "good" in one equilibrium and "bad" in the other. It follows that the possibility of negative prices cannot be avoided without introducing some particular assumptions.

satiation consumptions $\tilde{x}_i \in X_i$ (that is with $x_i \precsim_i \tilde{x}_i$ for every $x_i \in X_i$) satisfy the condition $\sum_{i=1}^{n} \tilde{x}_i \nleq \sum_{i=1}^{n} \omega_i$, or that there is at least one consumer (assuming that the consumption sets are closed and have a lower bound) whose preferences satisfy the global non satiation condition (introduced in Paragraph 3.2) .

Proof. If the price vector p is neither semipositive nor seminegative, then either all the prices are equal to zero or there is at least one negative and one positive price. In both cases the budget set of each consumer coincides with the consumption set. That is if $p \ngtr 0$ or $p \nless 0$, then $B_i(p) = \{x_i \in X_i : x_i \leq x_i'$ for some $x_i' \in \mathbb{R}^k$ such that $px_i' = p\omega_i\} = X_i$. In fact, if all the prices are equal to zero, then the budget set and the consumption set trivially coincide. If there are at least one negative and one positive price, then for every $x_i \in X_i$ the consumer can buy a bundle of goods x_i' such that $px_i' = p\omega_i$ and $x_i' \geq x_i$. If he buys enough of the good with negative price he will be able to buy any quantity of goods with positive price. In this way, using free disposal, he can acquire any $x_i \in X_i$. Therefore, if for the satiation consumptions $\tilde{x}_i$ the relationship $\sum_{i=1}^{n} \tilde{x}_i \nleq \sum_{i=1}^{n} \omega_i$ holds, then the feasibility condition (Definition 8.4) is not satisfied and the vector p is not an equilibrium price vector. By analogy, if a consumer has preferences that satisfy the global non satiation condition. In fact, assuming that his consumption set is closed and has a lower bound, the global non satiation condition implies that his consumption set is unbounded and he likes an infinite amount of at least one good. As a result, an unfeasible allocation . □

Proposition 11.2 allows to consider only semipositive price vectors in the analysis of the equilibrium with free disposal. In fact, on one hand, the clause "if the consumers are altogether sufficiently greedy" must always be considered satisfied. It holds if there is at least one consumer with globally non satiated preferences. If there are no consumers who satisfy the global non satiation condition and consumers were not altogether sufficiently greedy, then the economy would be a paradise, since all the consumers could obtain their satiation consumption, and there would not be any reason to use economic analysis. On the other hand, since the consumers are interested only in exchange ratios (considering the prices in the budget set only through the condition $px_i' = p\omega_i$) a seminegative price vector is equivalent to its opposite, for which we can limit to considering only semipositive vectors.

11.4 Existence of competitive equilibrium in pure exchange economy with free disposal

The existence of equilibrium means that the conditions that define it do not contradict one another, that is there exists a vector of prices that makes the choices of consumers feasible. The proof of the equilibrium existence requires some assumptions, for which we must qualify the economy $\mathcal{E} = (\langle X_i, \succsim_i \rangle, \omega_i, i = 1,\ldots, n)$ under examination. That is, the consumption sets X_i, systems of preferences $\langle X_i, \succsim_i \rangle$ and the endowments ω_i. Many of these assumptions were defined in Chapter 3 and will be recalled here.

Equilibrium existence is proved using fixed point theorems. The assumptions imposed on the economy $\mathcal{E}$ reflect assumptions required by these theorems. The two fixed point theorems relevant for the equilibrium with free disposal are the theorems of Brouwer and of Kakutani.

Brouwer Theorem: If $S \subset \mathbb{R}^k$ is a non-empty, compact (that is closed and bounded) and convex set and $f : S \to S$ is a continuous function, then there exists a fixed point, that is there is a $x^* \in S$ such that $x^* = f(x^*)$.[2]

Kakutani Theorem: If $S \subset \mathbb{R}^k$ is a non-empty, compact and convex set and ϕ: $S \to S$ is an upper hemicontinuous correspondence with convex image-sets $\phi(x)$ for all $x \in S$, than there exists a fixed point, that is there exits a $x^* \in S$ such that $x^* \in \phi(x^*)$.

Let us take under consideration the application of Brouwer theorem, which require that we deal with demand function (so we exclude demand correspondences in the following analysis). The continuity condition of Brouwer theorem requires that consumers' aggregate demand function, that is $D(p) = \sum_{i=1}^{n} d_i(p)$, is continuous and so is the aggregate excess demand function $E(p) = \sum_{i=1}^{n} e_i(p)$, where $e_i(p) = d_i(p) - \omega_i$ and so $E(p) = D(p) - \Omega$ (excess demand functions were introduced in Paragraph 4.5).

[2] The general proof of Brouwer theorem is not presented. The proof is, nevertheless, easy in one dimensional case, that is for $k = 1$, when set S is an interval.

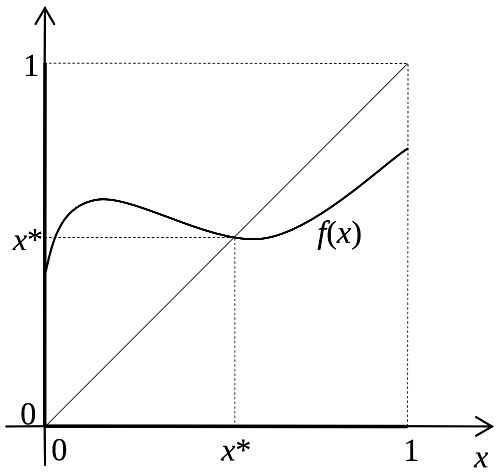

Figure 11.1

As shown in Figure 11.1, where $S = [0,1]$, a continuous function $f : [0,1] \to [0,1]$ has to have at least one point on the 45 degree line.

This condition can be introduced directly, without justification on the choice analysis basis, or sufficient conditions that guarantee continuity can be, highlighted therein, examined. In such a case, we have that the aggregate demand function is continuous if it is continuous for every consumer and this continuity is guaranteed, defining $m_i = p\omega_i$, by the Proposition 3.7, according to which demand function $d_i(p)$ (and therefore also excess demand function $e_i(p)$) is continuous if the budget set $B_i(p) = \{x_i \in X_i : px_i \leq p\omega_i\} = \{z_i \in Z_i : pz_i \leq 0\}$ (with $Z_i = X_i - \{\omega_i\}$) is non-empty, compact and convex, the correspondence $B_i : \mathbb{R}^k_+ \to X_i$ is continuous (and therefore also the correspondence $B_i : \mathbb{R}^k_+ \to Z_i$ is continuous), consumption set X_i is convex and the system of preferences $\langle X_i, \succsim_i \rangle$ is regular (that is complete and transitive), continuous and strictly convex.

Since X_i is non-empty and $\omega_i \in X_i$ (that is, Z_i is non-empty and $0 \in Z_i$), we get that the set $B_i(p)$ is non-empty for every p. The compactness of the set $B_i(p)$ is guaranteed if X_i (and so Z_i) is compact or if it is closed and has a lower bound (that is there exists a $\bar{x}_i \in \mathbb{R}^k$ such that $x_i \geq \bar{x}_i$ for every $x_i \in X_i$, as it is the case for example if $X_i = \mathbb{R}^k_+$) and $p >> 0$. The correspondence $B_i : \mathbb{R}^k_+ \to X_i$ (and therefore $B_i : \mathbb{R}^k_+ \to Z_i$) is continuous (by the Proposition 3.1) if X_i is compact, convex and $p\omega_i > \min_{x_i \in X_i} px_i$ (that is, $0 > \min_{z_i \in Z_i} pz_i$) for every $p > 0$.

Continuity condition of the correspondence $B_i : \mathbb{R}^k_+ \to X_i$ can create problems. For example, consider the case represented in Figure 11.2, where $X_i = \{x_i \in \mathbb{R}^2 : x_{i1} \in [0, a_{i1}], x_{i2} \in [0, a_{i2}]\}$ and $\omega_i = (\frac{1}{2} a_{i1}, 0)$. If we consider a sequence of price vectors that starts from $p = (p_1, p_2) >> 0$ and, by reducing only the price of the first good, tends to $p' = (0, p_2)$, we find out that the budget set $B_i(p_1, p_2) = \{x_i \in X_i : p_1 x_{i1} + p_2 x_{i2} \leq \frac{1}{2} p_1 a_{i1}\}$ becomes a smaller and smaller triangle, that is included in the previous sets. This triangle tends to the degenerate triangle $\lim_{p_1 \to 0} B_i(p_1, p_2) = \{x_{i1} \in [0, \frac{1}{2} a_{i1}], x_{i2} = 0\}$ represented by the segment $x_{i1} \in [0, \frac{1}{2} a_{i1}]$ on the horizontal axis. Instead, in correspondence to the vector $p' = (0, p_2)$, the budget set is $B_i(0, p_2) = \{x_{i1} \in [0, a_{i1}], x_{i2} = 0\}$, represented by the segment $x_{i1} \in [0, a_{i1}]$ on the horizontal axis. The inequality $\lim_{p_1 \to 0} B_i(p_1, p_2) \neq B_i(0, p_2)$ denotes discontinuity for the correspondence $B_i : \mathbb{R}^k_+ \to X_i$ in $p' = (0, p_2)$. The assumption which allows us to avoid this inconvenience is the one indicated above, that assumes that ω_i is an interior point to X_i (that is, 0 is interior to Z_i), so that we have $p\,\omega_i > \min_{x_i \in X_i} px_i$ (that is, $0 > \min_{z_i \in Z_i} pz_i$) for every $p > 0$.

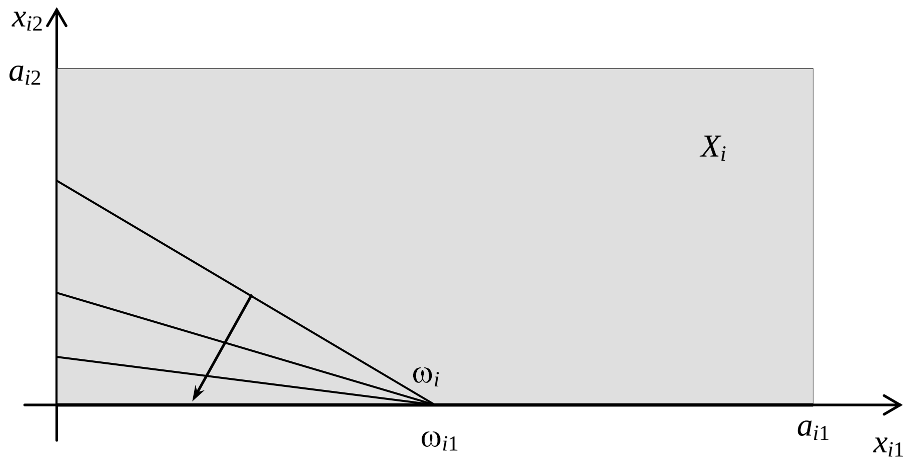

Figure 11.2

Moreover, if the system of preferences $\langle X_i, \succsim_i\rangle$ is monotone (or locally nonsatiated), then the demand function $d_i(p)$ satisfies the condition $p\ d_i(p) = p\ \omega_i$ (and the function $e_i(p)$ satisfies the condition $p\ e_i(p) = 0$) for every $p > 0$ (as indicated in the Proposition 3.4). As a result, we get the following proposition for the aggregate demand function (that derives from the Proposition 3.9).

Proposition 11.3 (*Walras Law*) If the systems of preferences of all consumers $\langle X_i, \succsim_i\rangle$, where $i = 1,\ldots, n$, are monotone, then for every $p > 0$ the relationship $p\ D(p) = p\ \Omega$ holds, where $\Omega = \sum_{i=1}^{n} \omega_i$, that is $p\ E(p) = 0$.

Walras law implies that if $p >> 0$ and there is equilibrium for k–1 goods (for example, we have $D_h(p) = \Omega_h$, that is $E_h(p) = 0$, for $h = 1,\ldots, k-1$), then there is equilibrium also for the k-th good (that is, $D_k(p) = \Omega_k$, and $E_k(p) = 0$).

Then, recalling Proposition 3.8, according to which demand functions are homogenous of degree zero, that is $d_i(\alpha p) = d_i(p)$ (and $e_i(\alpha p) = e_i(p)$) for every $\alpha > 0$, it is possible to normalize the prices. In fact, this property means that the choice of the consumer depends not on nominal prices but on exchange ratios (which are ratios between nominal prices), so it is possible modify the prices without changing ratios between one another and so without changing the demand of the consumers.[3] In other words, the demand functions do not depend on k variables (the number of nominal prices) but on k–1 variables (the number of relative prices). So, for example,

[3] Independence from the nominal prices reflects the fact that we can freely choose the accounting unit of the nominal prices. This unit, since it is an arbitrary choice of the observer, cannot have any impact on the choices of the consumers. Just like the choice of the unit of measurement of the distance from earth to the sun does not influence the amount of time needed for the earth to revolve around the sun (that also depends on this distance).

it is possible to describe prices by the following set (called k-1 dimensional *simplex*)

$$S^{k-1} = \{p \in \mathbb{R}^k_+ : \textstyle\sum_{h=1}^k p_h = 1\}$$

that determines all the possible relative prices of goods.[4] The Figure 11.3 and 11.4 indicate the simplexes, respectively, for $k = 2$ and for $k = 3$.

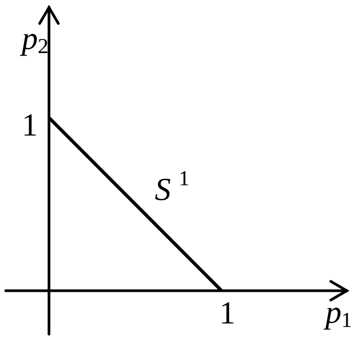

Figure 11.3

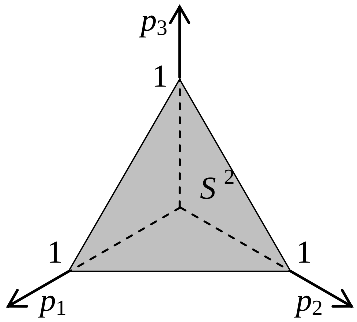

Figure 11.4

We note that the simplex is a non-empty, compact and convex set.

It is now possible to establish existence of equilibrium with the following proposition, that considers aggregate excess demand function $E : S^{k-1} \to Z$, where $Z = \sum_{i=1}^n Z_i$.

Proposition 11.4 (*Existence of the competitive equilibrium in pure exchange economy with free disposal*) There exists a $p^* \in S^{k-1}$ for which $E(p^*) \leq 0$ if $E : S^{k-1} \to Z$ (where Z is a compact subset of $\mathbb{R}^k$) is a continuous function such that $p\, E(p) = 0$ for every $p \in S^{k-1}$.

Proof. Let us introduce the function $G : S^{k-1} \to \mathbb{R}^{k-1}$ defined by the relationships

$$G_h(p) = \frac{p_h + \max\{0, E_h(p)\}}{1 + \sum_{h=1}^k \max\{0, E_h(p)\}} , \qquad h = 1,\ldots, k$$

We note that this function, which has domain S^{k-1}, has S^{k-1} as codomain (since $G_h(p) \in [0,1]$ for every $h = 1,\ldots, k$ and $\sum_{h=1}^k G_h(p) = 1$ for

[4] The indicated normalization considers a price $p_h' = \dfrac{p_h}{\sum_{h=1}^k p_h}$ in the place of price p_h for every h = 1,…, k. With this normalization, the cost of a bundle of goods that consists of one unit of each good is equal to 1 (that is, since the price of the bundle x is px, then the price of the bundle x = (1, 1,…, 1) is $\sum_{h=1}^k p_h$, which is necessarily positive since p is semipositive, so that we can normalize prices by choosing $\sum_{h=1}^k p_h$ =1). Other possible normalization can be obtained assuming that the price of a good is equal to 1 (called for this reason *numeraire*) under condition that in the equilibrium this price is not equal to 0. Another normalization, that can also be used also when negative prices are not excluded, considers the set $\{p \in \mathbb{R}^k : \sum_{h=1}^k p_h^{\,2} = 1\}$.

every $p \in S^{k-1}$) and is continuous since the function $E : S^{k-1} \to Z$ is continuous. Since, moreover, S^{k-1} is a non-empty, compact and convex set, then we can apply Brouwer theorem and we find that there exists a $p^* \in S^{k-1}$ such that $p^* = G(p^*)$. In relationship to this p^* we get

$$p_h^* = \frac{p_h^* + \max\{0, E_h(p^*)\}}{1 + \sum_{h=1}^{k} \max\{0, E_h(p^*)\}} \ , \qquad h = 1,\ldots, k$$

Multiplying by $E_h(p^*)$ $(1 + \sum_{h=1}^{k} \max\{0, E_h(p^*)\})$ both of the parts of this relationship, subtracting from both sides $p_h^* E_h(p^*)$ and summing the equalities obtained in this way with respect to h we get

$$\sum_{h=1}^{k} p_h^* \ E_h(p^*) \ \sum_{h=1}^{k} \max\{0, E_h(p^*)\} = \sum_{h=1}^{k} E_h(p^*) \ \max\{0, E_h(p^*)\}$$

and so, using Walras law,

$$0 = \sum_{h=1}^{k} E_h(p^*) \ \max\{0, E_h(p^*)\}$$

This relationship, given $E_h(p^*) \max\{0, E_h(p^*)\} \geq 0$ for every h = 1,…, k, implies $E_h(p^*) \max\{0, E_h(p^*)\} = 0$, i.e. $E_h(p^*) \leq 0$ for every h = 1,…, k. □

Propositions 11.3 and 11.4 imply the following proposition, according to which *free goods*, that is goods that are consumed in smaller quantity than is available, have zero prices. That is, if $E_h(p^*) < 0$, then $p_h^* = 0$.

Proposition 11.5 If the assumptions of Proposition 11.4 hold, then $E_h(p^*) < 0$ implies $p_h^* = 0$ and $p_h^* > 0$ implies $E_h(p^*) = 0$.

Proof. The conditions $p^* E(p^*) = 0$, $E(p^*) \leq 0$ and $p^* \in S^{k-1}$ imply $p_h^* E_h(p^*) = 0$ for every h = 1,…, k, from which the proposition derives. □

It is interesting to see under what conditions there are no free goods, that is $E(p^*) = 0$.

There are no free goods and, moreover, we have $p^* >> 0$ if there are some consumers with strongly monotone preferences and sufficiently large consumption sets. In fact, zero price would encourage these consumers to demand very high quantity of the good examined, equal to the maximum amount allowed by their consumption sets, and the total quantity demanded would be higher than the quantity available.

A condition (called *desirability condition*), which is weaker than strong monotonicity, is introduced by the following Definition 11.5. It ensures $E_h(p^*) = 0$ and $p_h^* > 0$ for a generic h-th good.

Definition 11.5 (*Desirability condition***)** A good is desirable if $E_h(p)$ is positive for every $p \in S^{k-1}$ with $p_h = 0$.

As a consequence $p_h^* > 0$ if h-th good is desirable. In fact, $p_h = 0$ excludes, if the desirability condition holds, that the feasibility condition $E_h(p) \leq 0$ be satisfied.

Note that the assumption of strong monotonicity of each consumer's preferences not only excludes that consumers use the possibility of free disposal (i.e. $px_i = p\omega_i$ for $x_i = d_i(p)$), but also requires (if consumption sets are sufficiently large) that the equilibrium allocation satisfies feasibility condition $\sum_{i=1}^{n} x_i \leq \sum_{i=1}^{n} \omega_i$ with equality (since $\sum_{i=1}^{n} px_i = \sum_{i=1}^{n} p\omega_i$ implies $\sum_{i=1}^{n} x_i = \sum_{i=1}^{n} \omega_i$ for $p >> 0$). As a result, if the preferences are strongly monotone, competitive equilibrium with free disposal coincides with equilibrium without free disposal (these equilibria were introduced in Definition 11.4).

The assumptions of Proposition 11.4 are sufficient but not necessary conditions for equilibrium existence (just like the conditions in the Brouwer theorem). A competitive equilibrium can therefore exist even if these conditions are not satisfied. (Also the conditions from the Proposition 3.7 are sufficient but not necessary for the continuity of individual demand functions). This reduces the relevance of the existence theorems. In fact, the logical reconstructions of economic reality operated by the general equilibrium theory (recall Paragraph 1.2) finds in the existence theorems the proof of the logical consistency of the theory, that is of feasibility of intentional choices. However, the logical truth of the theory, given that economics is an empirical science, is only a necessary condition in order for the theory to represent reality. That is logical truth is required by empirical truth but does not imply it. In other words, a logically false (i.e. contradictory) theory cannot represent reality, but a logically consistent theory may be empirically false. If the purpose of the theory is a logical and empirically true reconstruction of some economic reality, then the conditions of the existence theorems are sufficient in order to warrant a necessary condition. They are therefore non decisive conditions. In other words, it can happen that assumptions (of continuity of the aggregate excess function, etc.) are empirically satisfied but equilibria that they give rise to do not fit reality. On the contrary, it can happen that these conditions are not satisfied and nevertheless there exists an equilibrium that is a very good representation of reality. (If assumptions were necessary conditions, then their empirical falsification would imply both that the equilibrium relationships are contradictory and, consequently, that equilibrium is empirically false). Overall, even with this limitation, the proof of the existence of the competitive equilibrium was a great progress in economics as it allowed us to assess logical consistency of the general equilibrium theory under sufficiently weak conditions (essentially boiling down to the continuity of the aggregate excess demand function).

Competitive equilibria of a pure exchange economy can be computed by determining first the choices of the consumers (as shown in Paragraph 3.8, defining $m_i = p\omega_i$) and then imposing the feasibility condition $\sum_{i=1}^{n} x_i \leq \sum_{i=1}^{n} \omega_i$.

Exercise 11.1 Recall Exercise 4.1 where we considered, in an economy with two goods, an agent with a Cobb-Douglas utility function $u = x_1^{a} x_2^{1-a}$, where $a \in (0,1)$. Assume that there are n consumers and that they have a Cobb-Douglas utility function, so that the i-th consumer is represented by a utility function $u_i = x_{i1}^{a_i} x_{i2}^{1-a_i}$ and an endowment $\omega_i = (\omega_{i1}, \omega_{i2}) >> 0$. As a result we get individual demand functions

$$d_i(p_1, p_2) = \begin{bmatrix} a_i(\omega_{i1} + \omega_{i2} p_2 p_1^{-1}) \\ (1-a_i)(\omega_{i2} + \omega_{i1} p_1 p_2^{-1}) \end{bmatrix}, \qquad i = 1, \dots, n$$

and aggregate demand function

$$D(p_1, p_2) = \begin{bmatrix} \sum_{i=1}^{n} a_i(\omega_{i1} + \omega_{i2} p_2 p_1^{-1}) \\ \sum_{i=1}^{n} (1-a_i)(\omega_{i2} + \omega_{i1} p_1 p_2^{-1}) \end{bmatrix}$$

Keeping in mind that the preferences are strongly monotone so the feasibility condition can be satisfied as an equality, we obtain the following conditions

$$\sum_{i=1}^{n} a_i(\omega_{i1} + \omega_{i2} p_2 p_1^{-1}) = \sum_{i=1}^{n} \omega_{i1}\,, \qquad \sum_{i=1}^{n} (1-a_i)(\omega_{i2} + \omega_{i1} p_1 p_2^{-1}) = \sum_{i=1}^{n} \omega_{i2}$$

not independent (by Walras law). Each of them determines the equilibrium exchange ratio

$$\frac{p_2^*}{p_1^*} = \frac{\sum_{i=1}^{n} (1-a_i)\omega_{i1}}{\sum_{i=1}^{n} a_i \omega_{i2}}$$

Equilibrium allocation is determined by plugging this value in individual demand functions.

Now we assume, keeping other assumptions the same, that $n = 2$, $a_2 = 1$ (that is, $u_2 = x_{21}$), $\omega_1 = (0,1)$ and $\omega_2 = (2,1)$. The demand of the first consumer is then $x_{11} = a_1 p_2 p_1^{-1}$ for every $p \in S^1$ and $x_{12} = 1 - a_1$ for every $p \in S^1$ with $p_2 > 0$, while $x_{12} = \infty$ for $p_2 = 0$. The demand of the second consumer is $x_{21} = 2 + p_2 p_1^{-1}$ for every $p \in S^1$ and $x_{22} = 0$ for every $p \in S^1$ with $p_2 > 0$, while $x_{22} \in [0, \infty)$ (that is any point in the interval) for $p_2 = 0$. It is possible to show that these choices are not feasible. In fact if $p_2 > 0$, feasibility requires $a_1 p_2 p_1^{-1} + 2 + p_2 p_1^{-1} \le 2$, that is $(1 + a_1) p_2 p_1^{-1} \le 0$, condition that is never satisfied for $p_2 > 0$. If $p_2 = 0$, the choices are not feasible because the first consumer would demand an infinite quantity of the second good. In such a case no competitive equilibrium exists.

11.5 A graphical representation of pure exchange equilibrium with two goods and two agents: the Edgeworth-Pareto box diagram

The Edgeworth-Pareto box diagram (introduced in Paragraph 8.3 and Figure 8.1) provides a representation of economy $\mathcal{E} = (\langle X_1, \succsim_1 \rangle, \langle X_2, \succsim_2 \rangle, \omega_1, \omega_2)$. In what follows we assume that $X_1 = X_2 = \mathbb{R}^2_+$, $\omega_1, \omega_2 \in \mathbb{R}^2_+$ and that the systems of preferences of both consumers are regular, continuous and strongly monotone.

An economy of this type is a poor representation of the theory of general competitive equilibrium. On one hand, it is not very credible to assume that the two agents (only one buyer and only one seller of each of the two goods) are price-takers. (However, in this respect we could imagine that there are two types of agent and that the number of agents is equal for the two types and sufficiently large). On the other hand, the presence of only two goods (so only one exchange ratio) cannot generate that interdependence between markets which justifies general equilibrium versus partial equilibrium analysis. Nevertheless, the Edgeworth-Pareto box diagram is an instrument that is a very useful tool to understand many aspects of the general equilibrium.

In the Edgeworth-Pareto box in Figure 11.5 the initial allocation $\omega = (\omega_1, \omega_2)$ is represented by a point. Figure 11.5 contains all elements of economy $\mathcal{E} = (\langle X_1, \succsim_1\rangle, \langle X_2, \succsim_2\rangle, \omega_1, \omega_2)$. Now we will try to find competitive equilibria.

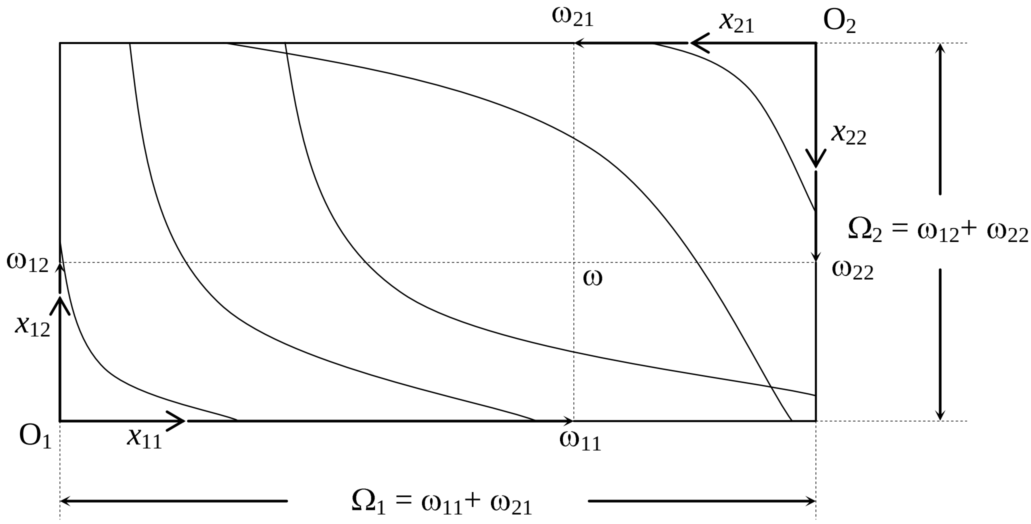

Figure 11.5

Since preferences are strongly monotone, the the two consumers choose, whatever the prices are, bundles of goods that satisfy budget constraints with equality. That is for any choice x_i, we have $p_1x_{i1}+p_2x_{i2} = p_1\omega_{i1}+p_2\ \omega_{i2}$, for $i = 1, 2$. This relationship is represented in the diagram by a line passing through point ω with a negative slope, the absolute value of which is equal to the exchange ratio p_1/p_2. In the diagram, as shown in Figure 11.6, the budget constraint lines of both consumers coincide. The difference comes from the fact that the budget line of the first consumer is read with respect to the origin O_1, and the one of the second consumer with respect to O_2. The choice of each consumer with respect to the possible values of the exchange ratio is depicted by price-consumption curve (introduced in Paragraph 3.8 and in Figure 3.16), which is a representation of the Walrasian demand function.

Feasibility condition, in presence of free disposal, requires that $x_{1h}+x_{2h} \leq \Omega_h$ (where $\Omega_h = \omega_{1h}+\omega_{2h}$), for $h = 1, 2$. With strongly monotone preferences, then, as shown in Paragraph 11.4 right after Proposition 11.5, there are no free goods and feasibility condition is satisfied with equality, that is $x_{1h}+x_{2h} = \Omega_h$, for $h = 1, 2$. As a result, feasible allocations are points in the set $C = \{x_1, x_2 \in \mathbb{R}^2_+ : x_1 + x_2 = \Omega\}$ that coincides in Edgeworth-Pareto box with the set of points in the rectangle.

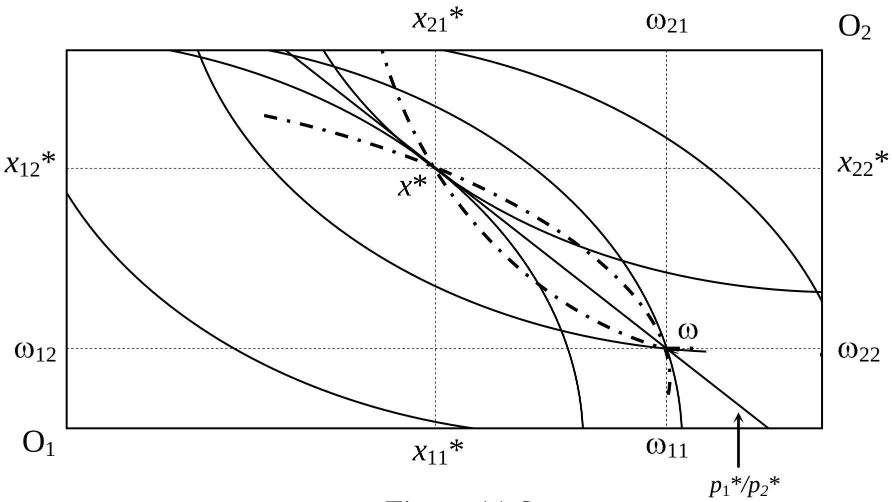

Figure 11.6

Competitive equilibrium requires that we find the bundles of goods that are, on one hand, chosen, that is belong to the price-consumption curves (represented in Figure 11.6 by dash curves), and, on the other hand, give rise to a feasible allocation, that is to bundles of goods represented for the two individuals by the same point in Edgeworth-Pareto box. As a consequence we get competitive equilibrium allocation only if it is represented by a point that belongs to both price-consumption curves. The further condition required for competitive equilibrium is that the choices of the consumers are determined by the same exchange ratio. This condition is satisfied by all points that price-consumption curves have in common, excepting possibly point ω. (If preferences are convex, then point ω belongs to the both price-consumption curves. However, it does not normally represent a competitive equilibrium allocation, because normally the two consumers have different marginal rates substitution at that point. It would be a competitive equilibrium allocation if they were equal). The points belonging to both price-consumption curves (except for point ω) represent choices with respect to the same exchange ratio, because they are on the same budget constraint line. We note that the indifference curves of the two individuals never intersect in the equilibrium point (they are tangent to each other if they are smooth). In Figure 11.6 we represent an economy with a unique competitive equilibrium allocation $x^* = (x_1^*, x_2^*)$. It is possible to draw cases without competitive equilibria (for example if preferences are not convex) or exhibiting multiple or infinite equilibria (for example if goods are perfect complements for both consumers).

From the Figure 11.6 we obtain a competitive equilibrium allocation $x^* = (x_1^*, x_2^*)$ that is efficient (Pareto optimal), as illustrated in Figure 8.1. In fact the indifference curves of the agents are tangent to each other in the point that represents competitive allocation, since they are both tangent in this point to the budget constraint line. This property, according to which competitive equilibrium allocations are efficient is remarkable. It is called "first welfare theorem" and will be presented for the general case of n agents and k goods in Paragraph 11.6. In this paragraph we will also present the "second welfare theorem", that is illustrated by the following property in Edgeworth-Pareto box. Take any efficient allocation (that is a point on the curve that is depicted in Figure 8.1). We find that this allocation can be obtained through competitive equilibrium if the preferences are convex (on the top of being continuous and strictly monotone), with appropriate endowment allocation. The Figure 11.7 depicts it with respect to two efficient allocations x^* and $\hat{x}$. Efficient allocation x^* is a competitive allocation if the initial endowment is ω^* (or some other point on the line tangent to indifference curves at x^*); the efficient allocation $\hat{x}$ is a competitive allocation if the initial endowments is $\hat{\omega}$ (or some other point on the tangent line). The slope of the line tangent to the indifference curves in the efficient allocation determines the exchange ratio of the corresponding competitive equilibrium.

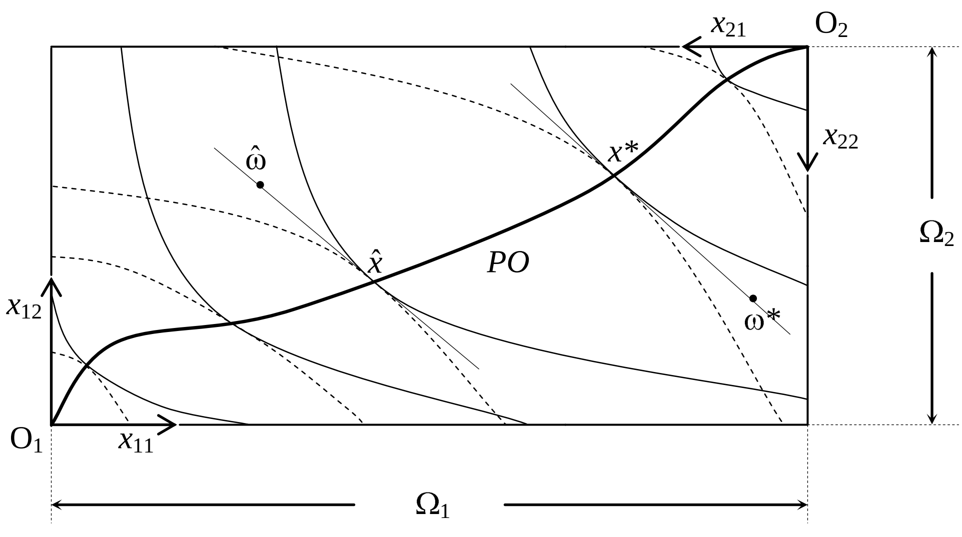

Figure 11.7

In Figure 11.8 we show some *non-competitive* equilibria, always for an economy with two consumers and two goods. Point x^* represents a competitive equilibrium allocation. (This is a point in which indifference curves of the two consumers are both tangent to the line that links this point to the endowment allocation point. Also the price-consumption curves go through this point. However in the figure, only the curve for consumer 1 is drawn). Point x_m represents the equilibrium allocation when consumer 2 is monopolist, that is the allocation that is obtained when this individual can choose the exchange ratio p_2/p_1 (this allocation is on the price-consumption curve of consumer 1, the one that is more convenient for consumer 2). Point x_{md} represents the allocation when there is monopoly with first-degree price discrimination, that is the equilibrium allocation that is obtained when the consumer 1 can only accept or reject the allocation proposed by consumer 2 (this allocation is the best point for consumer 2 on the indifference curve of consumer 1 determined by endowment ω_1). Note: *i*) allocations x^* and x_{md} are efficient and allocation x_m is inefficient; *ii*) individual 2 prefers allocation x_{md} to x_m and x_m to x^* (vice versa for consumer 1); *iii*) the quantity of good 2 sold by individual 2 in the monopoly case is smaller than in the competitive case and it is sold at a higher price, that is $(p_2/p_1)_m > (p_2/p_1)^*$, in accordance with the partial equilibrium theorem indicated in §10.8. Instead, contrary to what was obtained in the analysis of partial equilibrium (§10.10), the sold quantity of good 2 is not necessarily higher in the monopoly with first-degree price discrimination than in the pure monopoly and not necessarily its marginal price is smaller.

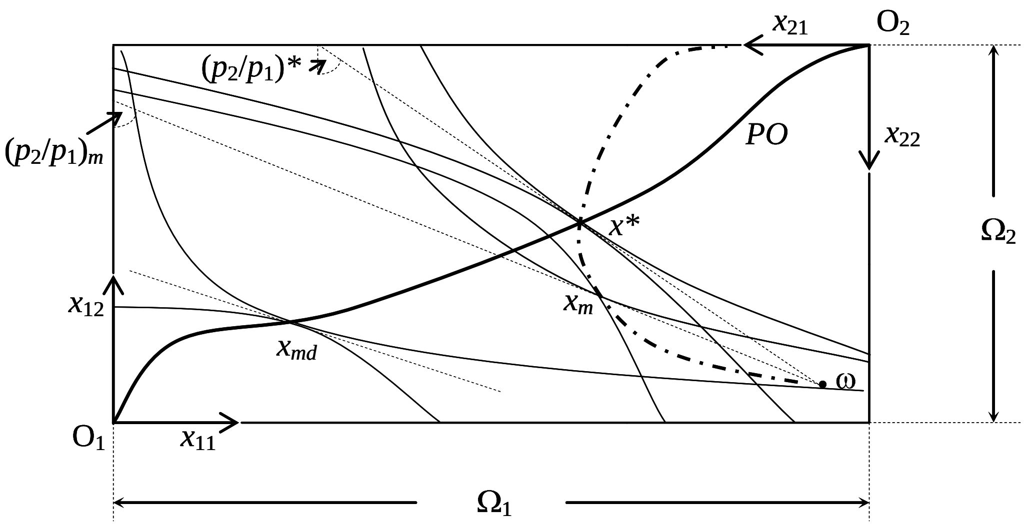

Figure 11.8

11.6 Efficiency of competitive equilibrium allocations in a pure exchange economy: the two welfare theorems

The competitive equilibrium in pure exchange economy with free disposal is represented by a price vector $p^* \in S^{k-1}$ and an allocation $x^* = (x_1^*,\ldots, x_n^*)$ such that $x_i^* \in d_i(p^*)$ for every $i = 1,\ldots, n$ (as indicated in Definition 11.4). We will now examine the relationship between allocations obtained in such a way and the efficient allocations (introduced in Paragraph 8.2) for a pure exchange economy. There are two principal propositions with regard to this topic. The first one states the conditions under which the competitive equilibrium allocation is efficient. The second one states the conditions under which the efficient allocation can be sustained by a competitive equilibrium.

Proposition 11.6 (*The First Welfare Theorem*) If (x*, p*) is a competitive equilibrium with free disposal for the economy (without externalities) $\mathcal{E} = (\langle X_i, \succsim_i \rangle, \omega_i, i = 1,\dots, n)$, then the allocation x* is weakly efficient.

Proof. Let's consider an equivalent proposition, according to which if an allocation is not weakly efficient, it cannot be a competitive equilibrium. If x is not a weakly efficient allocation, then there exists another allocation x' that is feasible, that is $\Sigma_{i=1}^{n} x_i' \leq \Sigma_{i=1}^{n} \omega_i$, such that $x_i' \succ_i x_i$ for every i = 1,…, n. As a result, for every $p \in S^{k-1}$ we get $\Sigma_{i=1}^{n} p x_i' \leq \Sigma_{i=1}^{n} p\omega_i$ and so there is at least one i = 1,…, n for which $px_i' \leq p\omega_i$. However, if there is, for every $p \in S^{k-1}$, some i = 1,…, n for which $px_i' \leq p\omega_i$ and $x_i' \succ_i x_i$, then $x_i \notin d_i(p)$ and so allocation x cannot constitute a competitive equilibrium. □

The preceding proposition, which is very general, requires some additional assumptions (other than absence of externalities) for strong efficiency of the competitive equilibrium. For example, competitive equilibrium is strongly efficient when the preferences of all the consumers are continuous and strongly monotone. In such a case, a weakly efficient allocation is, by Proposition 8.3, also strongly efficient.[5]

Nevertheless, it is possible to reformulate the first welfare theorem with respect to the strong efficiency without requiring the preferences to be continuous and strongly monotone. It is enough to assume that they are locally nonsatiated.

Proposition 11.7 If (x*, p*) is a competitive equilibrium (with *free disposal*) of an economy (without externalities) $\mathcal{E} = (\langle X_i, \succsim_i \rangle, \omega_i, i = 1,\dots, n)$, with regular (that is, complete and transitive) and locally nonsatiated preferences for all consumers, then the allocation x* is strongly efficient.

Proof. Assume, by contradiction, that (x*, p*) is a competitive equilibrium but the allocation x* is not strongly efficient. Then there exists an allocation x that is feasible, $\Sigma_{i=1}^{n} x_i \leq \Sigma_{i=1}^{n} \omega_i$, such that $x_i \succsim_i x_i^*$ for every i = 1,…, n and $x_j \succ_j x_j^*$ for at least one j = 1,…, n. Consequently, since $x_j^* \in d_j(p^*)$, we have for this consumer $p^* x_j > p^* \omega_j$ and for all the other consumers $p^* x_i \geq p^* \omega_i$ (otherwise, if $p^* x_i < p^* \omega_i$, because of locally nonsatiated preferences, there would exist in the neighborhood of x_i a point x_i' such that $x_i' \succ_i x_i \succsim_i x_i^*$ and $p^* x_i' < p^* \omega_i$, that is in

[5] Moreover, we keep in mind that this equivalence requires that the consumption sets are sufficiently large, for instance $X_i = \mathbb{R}_+^k$ for every i = 1,…, n.

contradiction to $x_i^* \in d_i(p^*)$). As a result, $\Sigma_{i=1}^n p^* x_i > \Sigma_{i=1}^n p^* \omega_i$, that is in contradiction to the feasibility condition $\Sigma_{i=1}^n x_i \leq \Sigma_{i=1}^n \omega_i$. □

The following example considers an economy that satisfies the assumptions from Proposition 11.6, but not the ones from Proposition 11.7 (that require the preferences to be locally nonsatiated). In this example there is a competitive equilibrium that presents a weakly but not strongly efficient allocation.

The example, represented in Edgeworth-Pareto box diagram in Figure 11.9, considers an economy with two consumers and two goods. The first consumer has not locally nonsatiated preferences (indicated by the thick indifference curve). Allocation x^* (coinciding with the endowment ω) is a competitive equilibrium allocation, where the prices are represented by the slope of the budget constraint shown on the figure. This allocation is weakly, but not strongly, efficient, because there are other feasible allocations, like allocation $\hat{x}$, that are preferred by the second consumer to x^* and leave the first consumer indifferent.

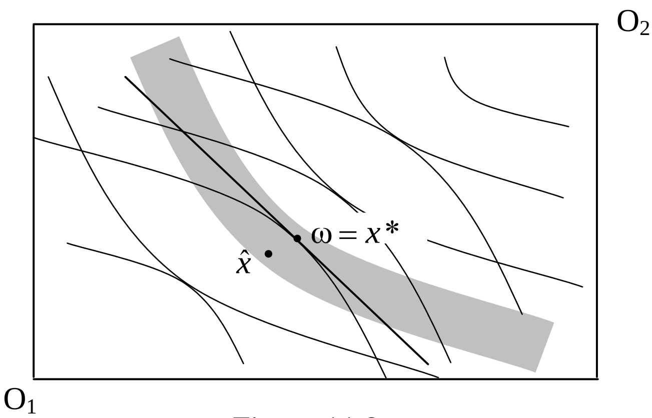

Figure 11.9

Proposition 11.6 requires that the first order conditions for equilibrium imply the first order conditions for efficiency (presented in the Paragraph 8.2), that is that the marginal rates of substitution are equal for each pair of goods for all the consumers. This is immediately verified, because the first order conditions of the choice of each consumer require the equality of every marginal rate of substitution to the exchange ratio of the two corresponding goods and competitive equilibrium requires that the prices are the same for all the consumers.

The main assumption in the Proposition 11.6 is the absence of externalities. If there are externalities, then, in general, we have a *market failure* (that is, the allocation generated in equilibrium is inefficient). In such a case, in order to reach efficiency we must introduce corrections, which can have the form of taxes or subsidies on prices for the goods that create externalities or also, if the externalities are produced by the firms, the form of mergers. These aspects are analyzed in Paragraph …. Another assumption, set out by the Proposition 11.6, is the common knowledge of the characteristics of the goods exchanged. If there is asymmetric information (for example some agents discriminate between goods because they have information on their quality that other agents have not), then the competitive equilibrium allocation can be inefficient. This problem (known as *adverse selection*) is analyzed in Paragraph …

The *Second Welfare Theorem* looks at the possibility of arriving at some assigned efficient allocation through competitive equilibrium (with an appropriate endowment allocation). This possibility requires rather restrictive conditions, among which the most important is the convexity of the preferences. Among the possible formulations of this theorem, one of the simplest ones is the following.

Proposition 11.8 (*Second Welfare Theorem*) Let x^* be an efficient allocation in the economy (without externalities) $(\langle X_i, \succsim_i\rangle, \Omega, i = 1,\dots, n)$. If, for every $i = 1,\dots, n$, the system of preferences $\langle X_i, \succsim_i\rangle$, where $X_i = \mathbb{R}^k_+$, is regular (that is complete and transitive), continuous, strongly monotone[6] and convex and $x_i^* >> 0$, than there exists a vector $p^*\in S^{k-1}$ for which (x^*, p^*) is a competitive equilibrium for an economy $\mathcal{E} = (\langle X_i, \succsim_i\rangle, \omega_i, i = 1,\dots, n)$, where the endowments ω_i are such that $\Sigma^n_{i=1}\omega_i = \Omega$ and $p^*\omega_i = p^*x_i^*$ for every $i = 1,\dots, n$ (for example if $\omega_i = x_i^*$ for every $i = 1,\dots, n$).

Proof. The proof is rather lengthy so we are going to break it up in the following steps.

a) We consider, for every consumer, the set of bundles that are preferred to x_i^*, that is $P_i(x_i^*) = \{x_i\in\mathbb{R}^k_+ : x_i \succ_i x_i^* \}$, and their sum $P(x^*) = \Sigma^n_{i=1}P_i(x_i^*) = \{X\in\mathbb{R}^k_+ : \Sigma^n_{i=1}x_i = X \text{ and } x_i \succ_i x_i^* \text{ for every } i = 1,\dots, n\}$. All these sets are convex, because the preferences of the consumers are convex and the sum of convex sets is a convex set.

b) Since allocation x^* is efficient, we have that $X^*\notin P(x^*)$, where $X^* = \Sigma^n_{i=1}x_i^*$. Therefore, since $X^*\in\mathbb{R}^k$ does not belong to $P(x^*) \subset \mathbb{R}^k$, we can apply the separating hyperplane theorem,[7] according to which there exists a

[6] The assumption that requires strongly monotone preferences can be substituted by the weaker assumption that preferences are locally nonsatiated. In such a case the proof has to be slightly modified.

[7] This theorem belongs to the family of separating hyperplane theorems. The theorem relevant for the examined proposition states that if a set $P \subset \mathbb{R}^k$ is convex and a point $x^*\in\mathbb{R}^k$ does not belong to it, that is $x^*\notin P$, then there exists a vector $a \neq 0$ such that $ax \geq ax^*$ for every $x\in P$. In Figure 11.10 we represent a case in which P is an open convex set and x^* belongs to its border.

vector $a \neq 0$ such that $aX \geq aX^*$ for every $X \in P(x^*)$. On the other hand, since the preferences are strongly monotone, the efficient allocation x^* satisfies the condition $\Sigma_{i=1}^{n} x_i^* = \Omega$. As a consequence $aX^* = a\Omega$.

c) Let's consider an allocation $(x_i^* + \frac{1}{n} e_h)_{i=1}^{n}$, where e_h is a vector with all the components equal to zero except the h-th component which is equal to 1. Since the preferences of all the consumers are strongly monotone we have $(x_i^* + \frac{1}{n} e_h) \succ_i x_i^*$ for all $i = 1, \ldots, n$ and, so, having $(X^* + e_h) = \Sigma_{i=1}^{n} (x_i^* + \frac{1}{n} e_h)$ we get $(X^* + e_h) \in P(x^*)$ for every $h = 1, \ldots, k$. As a result, applying the inequality from the separating hyperplane theorem, we get $a(X^* + e_h) \geq aX^*$, that is $a_h \geq 0$ for every $h = 1, \ldots, k$. Thus, since $a \neq 0$, we have $a > 0$. At this point we define $p^* = \frac{1}{\Sigma_{h=1}^{k} a_h} a$. We. therefore, get that $p^* \in S^{k-1}$ and $p^*X \geq p^*X^*$ for every $X \in P(x^*)$.

d) Now we will prove, for every $i = 1, \ldots, n$, that if $x_i \succ_i x_i^*$, then $p^*x_i \geq p^*x_i^*$. If $x_i \succ_i x_i^*$, since preferences are continuous and $X_i = \mathbb{R}_{+}^{k}$, then there exists a $t \in (0, 1)$ such that $(1 - t)\, x_i \succ_i x_i^*$. Let's consider an allocation x', with $x_i' = (1 - t)\, x_i$ and $x_b' = x_b^* + \frac{t}{n-1} x_i$ for every $b \neq i$ and $b = 1, \ldots, n$. Since the preferences are strongly monotone we get $x_i' \succ_i x_i^*$ for every $i = 1, \ldots, n$. Then we have $X' = \Sigma_{i=1}^{n} x_i' \in P(x^*)$ and so by the separating hyperplane theorem $p^* X' \geq p^*X^*$, that is $p^* \Sigma_{i=1}^{n} x_i' \geq p^* \Sigma_{i=1}^{n} x_i^*$. Keeping in mind the definition of the allocation x', we obtain

$$p^* (1 - t)\, x_i + p^* \Sigma_{b=1, b \neq i}^{n} \left(x_b^* + \frac{t}{n-1} x_i \right) \geq p^* \Sigma_{b=1}^{n} x_b^*$$

and as a result $p^*x_i \geq p^*x_i^*$.

e) It is possible to reinforce the preceding relation by proving, for every $i = 1, \ldots, n$, that if $x_i \succ_i x_i^*$, then $p^*x_i > p^*x_i^*$. In fact, as already shown, if $x_i \succ_i x_i^*$, since preferences are continuous and $X_i = \mathbb{R}_{+}^{k}$, then there exists a $t \in (0, 1)$ such that $(1 - t)\, x_i \succ_i x_i^*$. Then, applying the relationship determined

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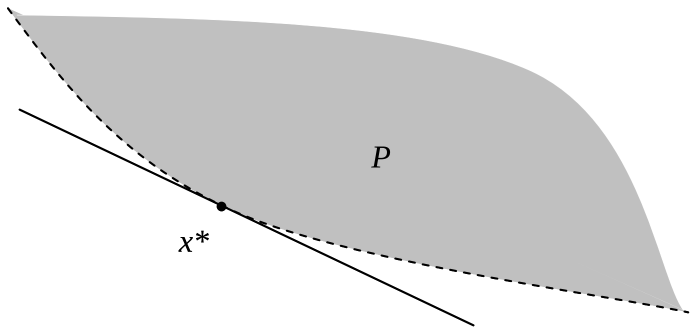

Figure 11.10

in the previous step, we get $p^*(1-t)\,x_i \geq p^*x_i^*$, that is $p^*x_i \geq \frac{1}{1-t}\,p^*x_i^*$. Since $p^* \in S^{k-1}$ and $x_i^* >> 0$ by assumption, it is $p^*x_i^* > 0$, and so the previous inequality requires $p^*x_i > p^*x_i^*$.

f) From the previous steps we get that for every $i = 1,\ldots, n$, if $x_i \succ_i x_i^*$, then $p^*x_i > p^*x_i^*$. This implies that if $p^*x_i \leq p^*x_i^*$, then $x_i \precsim_i x_i^*$ and so $x_i^* \succsim_i x_i$ for every $x_i \in \{x_i \in X_i\colon p^*x_i \leq p^*x_i^*\}$. As a consequence, for every $i = 1,\ldots, n$, that $x_i^* \in d_i(p^*, p^*\omega_i)$ for every ω_i such that $p^*\omega_i = p^*x_i^*$. Moreover, because x^* is a feasible (since efficient) allocation, we get that (x^*, p^*) is a competitive equilibrium. (We finally note that this result and the assumption that the preferences are strongly monotone imply also that $p^* >> 0$). □

In Figures 11.11 and 11.12 we present Edgeworth-Pareto diagrams corresponding to two cases in which efficient allocation cannot be obtained through a competitive equilibrium. In Figure 11.11 the reason is that preferences are not convex. In Figure 11.12 the bundle of goods x_1^* is not positive. (Figure 11.12 represents a situation where $u_1 = x_{11} + \sqrt{x_{12}}$ and $u_2 = \min\{x_{21}, x_{22}/2\}$, with $x_1^* = (1/2, 0)$ and $x_2^* = (1/2, 1)$).

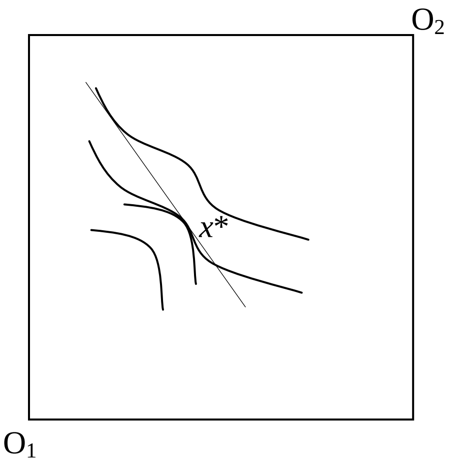

Figure 11.11

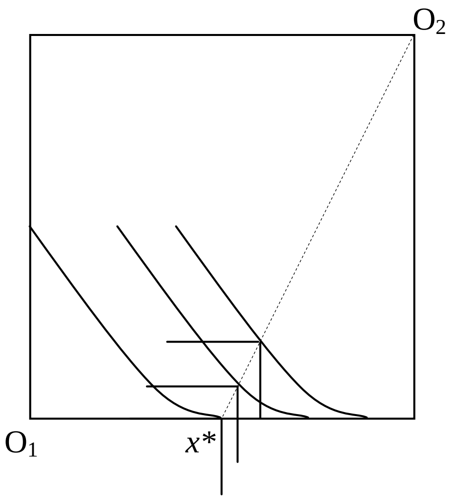

Figure 11.12

The second welfare theorem demonstrates that it is possible to obtain every efficient allocation (and so the allocation with maximum social welfare) with an appropriate distribution of resources in the competitive economy. This possibility is one of the fundaments of the policy for the reduction of inequality among individuals. Imposing taxes on the exchange introduces a distortion, because due to the resulting gap between the sell and buy price the marginal rates of substitution will not be equalized, which is a condition for efficiency. Introducing wealth transfers between the consumers does not create this problem. Nevertheless, in order to achieve a specific efficient allocation the transfers have to be such that, under competitive equilibrium, they give rise to the desired allocation. This requires that the authority that determines these transfers needs to know all the data about the economy $\mathcal{E}$ = $(\langle X_i, \succsim_i\rangle, \omega_i, i=1,\ldots, n)$. Lack of this knowledge is one of the main limits of the redistribution policy.

11.7 Competitive equilibrium and social welfare maximum

The two welfare theorems regard the relationship between competitive equilibrium and efficiency. Now, it is worthwhile to examine the relationship between the competitive equilibrium and social welfare maxima. Until now, we studied the relationship between efficiency and social welfare maxima in Paragraph 8.5. We found out that the allocation that maximizes social welfare function is efficient (Proposition 8.9) and that every efficient allocation maximizes at least one social welfare function (Proposition 8.10). We will now study whether the allocations that maximize social welfare can be obtained by competitive equilibrium and also if there exists a social welfare function that would be maximized by the allocation obtained under competitive equilibrium.

Proposition 11.9 With respect to the economy (without externalities) $(\langle X_i, \succsim_i\rangle, \Omega, i = 1,\ldots, n)$, let $x^* = (x_i^*)_{i=1}^n$ be an allocation that maximizes a given social welfare function $W(u_1(x_1),\ldots, u_n(x_n))$ over the set of feasible allocations $C_{FD} = \{(x_i)_{i=1}^n : x_i \in X_i,\ \sum_{i=1}^n x_i \leq \Omega\}$. If, for every $i = 1,\ldots, n$, the system of preferences $\langle X_i, \succsim_i\rangle$, where $X_i = \mathbb{R}_+^k$, is regular (that is complete and transitive), continuous, strongly monotone and convex and $x_i^* >> 0$, then there exists a vector $p^* \in S^{k-1}$ for which (x^*, p^*) is a competitive equilibrium of the economy $\mathcal{E} = (\langle X_i, \succsim_i\rangle, \omega_i, i = 1,\ldots, n)$, where the endowments ω_i are such that $\Sigma_{i=1}^n \omega_i = \Omega$ and $p^*\omega_i = p^*x_i^*$ for every $i = 1,\ldots, n$ (for example, if $\omega_i = x_i^*$ for every $i = 1,\ldots, n$).

Proof. The proof follows directly from Propositions 8.9 and 11.8. □

Proposition 11.10 If (x^*, p^*) is a competitive equilibrium, with free disposal, of the economy (without externalities) $\mathcal{E} = (\langle X_i, \succsim_i\rangle, \omega_i, i=1,\ldots, n)$, then the competitive allocation x^* maximizes at least one social welfare function. In particular, if preferences can be represented with utility functions $u_i(x_i)$ that are concave and monotone for every $i = 1,\ldots, n$, then the competitive allocation x^* maximizes the social welfare function $\Sigma_{i=1}^n \frac{1}{\lambda_i} u_i(x_i)$ over the set $C_{FD} = \{(x_i)_{i=1}^n : x_i \in X_i,\ \sum_{i=1}^n x_i \leq \sum_{i=1}^n \omega_i\}$, where λ_i is the indirect marginal utility of wealth of the i-th consumer, that is $\lambda_i = \mathrm{D}_{m_i} u_i^*(p^*, p^*\omega_i)$ for every $i = 1,\ldots, n$.

Proof. The first part of the proposition follows directly from the Propositions 8.10 and 11.6. We obtain the second part by proving that maximizing the proposed social welfare function leads exactly to the competitive equilibrium allocation. In fact, the Lagrangian for the problem $\max_{x \in C_{FD}} \Sigma_{i=1}^{n} \frac{1}{\lambda_i} u_i(x_i)$, is

$$L(x_1,\ldots, x_n, \mu_1,\ldots, \mu_k) = \Sigma_{i=1}^{n} \frac{1}{\lambda_i} u_i(x_i) + \Sigma_{h=1}^{k} \mu_h (\Sigma_{i=1}^{n} \omega_{ih} - \Sigma_{i=1}^{n} x_{ih})$$

We obtain the following first order conditions

$$\frac{1}{\lambda_i} \mathrm{D}_{x_{ih}} u_i(x_i) = \mu_h \quad \text{for every } i = 1,\ldots, n \text{ and every } h = 1,\ldots, k,$$

$$\Sigma_{i=1}^{n} \omega_{ih} - \Sigma_{i=1}^{n} x_{ih} = 0 \quad \text{for every } h = 1,\ldots, k.$$

The solution (x, μ) of these equations coincides with the competitive equilibrium (x^*, p^*) because they coincide with the first order conditions of the competitive equilibrium, since $\lambda_i = \mathrm{D}_{m_i} u_i^*(p^*, p^*\omega_i)$ for every i = 1,…, n. In fact, the first order conditions of the competitive equilibrium are the equations $\mathrm{D}_{x_{ih}} u_i(x_i) = \lambda_i p_h$, $\Sigma_{i=1}^{n} \omega_{ih} - \Sigma_{i=1}^{n} x_{ih} = 0$, for every i = 1,…, n and h = 1,…, k, and the budget constraints $px_i = p\omega_i$, for every i = 1,…, n, which imply $\lambda_i = \mathrm{D}_{m_i} u_i^*(p^*, p^*\omega_i)$ for every i = 1,…, n. The assumption that utility functions are concave, on one hand, implies that the second order conditions are satisfied, and, on the other hand, that the first order conditions determine the global maximum of social welfare. □

The social welfare function $W = \Sigma_{i=1}^{n} \frac{1}{\lambda_i} u_i(x_i)$ introduced in the previous proposition is a weighted average of the utilities of all the consumers. Since the utilities are concave, the weight $\frac{1}{\lambda_i}$ associated to the generic i-th consumer is larger the larger his wealth is.[8] In this way, the social welfare function keeps track of the differences in wealth between the consumers in order to generate, through its maximization, an allocation in which the consumers with larger wealth obtain richer bundles of goods.

[8] In fact, concavity implies that $\frac{\partial \lambda_i}{\partial m_i} = \mathrm{D}^2_{m_i m_i} u_i^*(p^*, p^*\omega_i) < 0$. This inequality is a result of the relationship obtained in the proof of the Proposition 3.13, that is $d\lambda^*$ = $-\frac{p^T (\mathrm{D}^2 u)^{-1}}{p^T (\mathrm{D}^2 u)^{-1} p} \lambda^* \mathrm{d}p - \frac{1}{p^T (\mathrm{D}^2 u)^{-1} p}(x^{*T} \mathrm{d}p - \mathrm{d}m)$, from which $\frac{\partial \lambda^*}{\partial m} = \frac{1}{p^T (\mathrm{D}^2 u)^{-1} p}$, which is positive if the utility function is concave. Notice that we require that the indirect utility function is convex with respect to wealth and not only quasi-convex (as guaranteed by Proposition 3.10).

11.8 Competitive equilibrium existence in production economy with free disposal

A private ownership production economy is represented by $\mathcal{E} = (\langle X_i, \succsim_i\rangle, Y_j, \omega_i, \theta_{ij}, i = 1,\ldots, n, j = 1,\ldots, m)$, just like in Paragraph 11.1. Profits belong to the consumers and depend for every consumer on the amount of shares he owns in every firm. Therefore, if there is free disposal, the budget constraint of the i-th consumer is

$$B_i(p,(\pi_j)_{j=1}^m) = \{x_i \in X_i : x_i \le x_i' \text{ for some } x_i' \text{ such that } px_i' = p\omega_i + \textstyle\sum_{j=1}^m \theta_{ij}\pi_j\}$$

This set is a non decreasing correspondence with respect to profits (that is, $\pi_j' \le \pi_j$ for $j = 1,\ldots, m$ implies $B_i(p,(\pi_j')_{j=1}^m) \subseteq B_i(p,(\pi_j)_{j=1}^m)$). Then, none of the consumers is against profit maximization, because choosing from a larger set cannot lead to a less preferred choice. Therefore, the assumption that firms maximize profit is justified.

Proposition 11.2, that holds also for production economies (if $0\in Y_j$ for every $j = 1,\ldots, m$, as it is easily proved), implies that we can consider only semipositive price vectors in the analysis of the competitive equilibrium with free disposal. This implication can also be obtained from the free disposal condition for production. If free disposal condition holds for at least one firm, then there exists a j such that $Y_j - \mathbb{R}_+^k \subset Y_j$, by which if $0\in Y_j$, then $-\mathbb{R}_+^k \subset Y_j$. If there was a good with a negative price, then maximizing profit of this firm would lead to an infinite demand for this good and an infinitely large profit, and as a result equilibrium inexistence. (We note that this reasoning holds if the set Y_j is not bounded). Therefore, with a semipositive price vector $p > 0$, the above budget set of i-the consumer becomes

$$B_i(p,(\pi_j)_{j=1}^m) = \{x_i \in X_i :\ px_i \le p\omega_i + \textstyle\sum_{j=1}^m \theta_{ij}\pi_j\}.$$

The choice of every firm is derived from its profit maximization over its production set and it is represented by a supply function $s_j(p)$. The choice of every consumer is derived from maximization of his utility over his budget set. The feasibility condition, in presence of free disposal, requires $\sum_{i=1}^n x_i \le \sum_{j=1}^m y_j + \sum_{i=1}^n \omega_i$. Therefore, equilibrium is described by an allocation and a vector of prices, that is by $((x_i^*)_{i=1}^n, (y_j^*)_{j=1}^m, p^*)$, such that

$$y_j^* \in \{y_j \in Y_j : p^* y_j \ge p^* y_j' \text{ for every } y_j' \in Y_j\}, \qquad j = 1,\ldots,m,$$

$$x_i^* \in \{x_i \in B_i(p^*,(p^* y_j^*)_{j=1}^m) : x_i \succsim_i x_i' \text{ for every } x_i' \in B_i(p^*,(p^* y_j^*)_{j=1}^m)\}, \qquad i = 1,\ldots,n,$$

$$\sum_{i=1}^n x_i^* \le \sum_{j=1}^m y_j^* + \sum_{i=1}^n \omega_i .$$

If the aggregate excess demand function

$$E(p) = \textstyle\sum_{i=1}^n (d_i(p) - \omega_i) - \sum_{j=1}^m s_j(p)$$

where $p \in S^{k-1}$ and, for every j = 1,…, m and i = 1,…, n,

$$s_j(p) \in \; y_j \in Y_j : py_j \geq py_j' \text{ for every } y_j' \in Y_j \;\; ,$$

$$\pi_j(p) = \max_{y_j \in Y_j} py_j ,$$

$$d_i(p) = \; x_i \in B_i(p, (\pi_j(p))_{j=1}^m) : x_i \succsim_i x_i' \text{ for every } x_i' \in B_i(p, (\pi_j(p))_{j=1}^m) \;\; ,$$

satisfies conditions stated in Proposition 11.4, then the last one ensures that the equilibrium exists.

Among the conditions required by Proposition 11.4, homogeneity of degree zero of the aggregate excess demand function and Walras law are surely satisfied. In fact, supply functions, whenever they are defined, are homogenous of degree zero (Proposition 5.3), the profit functions are homogenous of degree one (Proposition 5.3) and demand functions are, as a consequence, homogenous of degree zero (applying Proposition 3.8). Therefore, with the conditions imposed by Proposition 11.3, Walras law holds. The most problematic condition from Proposition 11.4, especially in the case of production, is the one that requires aggregate excess demand functions to be single-valued (that is to be proper functions and not correspondences) and, moreover, continuous. In fact, we must introduce strict convexity of the production set in order to have continuous supply function that takes only one value. Now, strict convexity excludes constant returns to scale (and allows only for decreasing returns to scale, that is λy_j is in the interior of Y_j if $0 \in Y_j$, $y_j \in Y_j$ and $\lambda \in (0,1)$).

If we want to prove equilibrium existence for an economy with production that exhibits non increasing (that is, also constant) returns to scale, then we cannot use Brouwer theorem (that we used to prove Proposition 11.4) and we have to use Kakutani theorem instead.

Let $Z = \sum_{i=1}^n X_i - \sum_{j=1}^m Y_j - \{\Omega\}$ is a compact and convex subset of $\mathbb{R}^k$. The proof of competitive equilibrium existence can be obtained (as in Proposition 11.11) if the aggregate excess demand correspondence $E : S^{k-1} \to Z$ is upper hemicontinuous and such that the set $E(p)$ is non-empty and convex. Moreover, budget constraints imply that $pE(p) \leq 0$ for every $p \in S^{k-1}$ (where $pE(p) \leq 0$ means that $pz \leq 0$ for every $z \in E(p)$), i.e. that the so-called *weak Walras law* holds (whilst Walras law requires $pE(p) = 0$)..

The set Z is compact if the sets X_i, for i = 1,…, n, and Y_j, for j = 1,…, m are compact. (Equilibrium existence can be proved also if these sets are not bounded. An interested reader can find the proof in Debreu, 1959, pp. 83-88. Moreover, boundedness of the sets can be justified if we recall that the quantity of the goods is in any case bounded and assume that the agents know it and reflect this knowledge in the corresponding consumption and production sets).

Having $E(p) = \sum_{i=1}^{n}(d_i(p) - \omega_i) - \sum_{j=1}^{m} s_j(p)$, the correspondence $E : S^{k-1} \rightarrow Z$ is upper hemicontinuous and the set $E(p)$ is non-empty and convex for every $p \in S^{k-1}$ if these characteristics are shared by the correspondences $d_i : S^{k-1} \rightarrow X_i$, for every $i = 1,\ldots, n$, and $s : S^{k-1} \rightarrow Y$ (we can refer directly to the aggregate supply correspondence one as shown in Paragraph 5.7). Now, for every $i = 1,\ldots, n$, the correspondence $d_i : S^{k-1} \rightarrow X_i$ is upper hemicontinuous and the set $d_i(p)$ is non-empty and convex for every $p \in S^{k-1}$ if the set X_i is non-empty, compact and convex, $p\omega_i + \sum_{j=1}^{m} \theta_{ij}\pi_j > \min_{x_i \in X_i} px_i$ for every $p \in S^{k-1}$ (this condition is satisfied if ω_i is in the interior of X_i and $0 \in Y_j$ for every $j = 1,\ldots, m$) and the system of preferences $\langle X_i, \succsim_i \rangle$ is regular, continuous and weakly convex (as we deduce from Propositions 3.3, 3.5 and 3.7). The correspondence $s : S^{k-1} \rightarrow Y$ is upper hemicontinuous and the set $s(p)$ non-empty and convex for every $p \in S^{k-1}$ if the set Y is non-empty, compact and convex (as we can deduce from Proposition 5.2, keeping in mind that these conditions allow to determine $s(p)$ for every $p \in S^{k-1}$). Finally, since $d_i(p) \subset B_i(p, (\pi_j)_{j=1}^{m})$ for every $i = 1,\ldots,n$ and $B_i(p, (\pi_j)_{j=1}^{m}) = \{x_i \in X_i : px_i \leq p\omega_i + \sum_{j=1}^{m} \theta_{ij}\pi_j\}$, then $px_i \leq p\omega_i + \sum_{j=1}^{m} \theta_{ij}\pi_j = p\omega_i + \sum_{j=1}^{m} \theta_{ij} py_j$ for every $x_i \in d_i(p)$, $y_j \in s_j(p)$ and $p \in S^{k-1}$. The sum of these inequalities with respect to $i = 1,\ldots,n$ leads to $pE(p) \leq 0$ for every $p \in S^{k-1}$ (i.e. $pz \leq 0$ for every $z \in E(p)$), which is the weak Walras law.

The assumptions that we considered are sufficient in order for the aggregate excess demand function $E :\ {}^{k}S^{1} \rightarrow$ to be upper hemicontinuous and for the set $E(p)$ to be non-empty, convex and such that $pE(p) \leq 0$ for every $p \in S^{k-1}$. Therefore : [9]

a) for every consumer $i = 1,\ldots,n$, the consumption set X_i is non-empty, compact and convex, the system of preferences $\langle X_i, \succsim_i \rangle$ is regular, continuous, weakly convex and ω_i is a point in the interior of X_i;

b) for the producers, Y is non-empty, compact and convex, such that $Y \cap \mathbb{R}^k_+ = \ 0$, with $0 \in Y_j$ for every $j = 1,\ldots,m$.

[9] Debreu (1959, pp. 82-88) proves equilibrium existence without assuming that consumption and production sets are bounded. In this case the following assumptions are required, for the consumers. X_i is non-empty, closed, bounded from below and convex. $\langle X_i, \succsim_i \rangle$ is regular, continuous, convex and without global satiation. ω_i is an interior point in X_i for every $i = 1,\ldots,n$. The following assumptions are required for the producers. Y is non-empty, closed and convex, such that $Y \cap (-Y) = \ 0$ and $-\mathbb{R}^k_+ \subset Y$, with $0 \in Y_j$ for every $j = 1,\ldots, m$.

Now, we can introduce Proposition 11.11 that proves the existence of competitive equilibrium in production economy with free disposal.

Proposition 11.11 (*Competitive equilibrium existence in production economy with free disposal*). If the aggregate excess demand correspondence $E: S^{k-1} \to Z$, where $Z = \sum_{i=1}^{n} X_i - \sum_{j=1}^{m} Y_j - \{\Omega\}$ is a compact and convex subset of $\mathbb{R}^k$, is upper hemicontinuous and such that the set $E(p)$ is non-empty, convex and satisfies the weak Walras law $pE(p) \leq 0$ for every $p \in S^{k-1}$, then there exists a $p^* \in S^{k-1}$ for which $E(p^*) \cap (-\mathbb{R}_+^k) \neq \varnothing$.

Proof. Let's introduce the correspondence $P: Z \to S^{k-1}$, with $P(z) = \arg\max_{p \in S^{k-1}} pz$. Since the set S^{k-1} is non-empty and compact, $P(z)$ is non-empty and convex for every $z \in Z$ and the correspondence $P: Z \to S^{k-1}$ is upper hemicontinuous (substantially, by Weirstrass and "maximum" theorems shown in Proposition 3.6 and 3.7 and corresponding comments). Let's now consider the correspondence $\phi: S^{k-1} \times Z \to Z \times S^{k-1}$ defined by $\phi(p,z) = E(p) \times P(z)$. The set $S^{k-1} \times Z$ (that coincides with the set $Z \times S^{k-1}$) is non-empty, compact and convex because Z and S^{k-1} are non-empty, compact and convex. The set $\phi(p,z)$ is non-empty and convex for every $(p,z) \in S^{k-1} \times Z$ because the sets $E(p)$ for every $p \in S^{k-1}$ and $P(z)$ for every $z \in Z$ are non-empty and convex. The correspondence $\phi: S^{k-1} \times Z \to Z \times S^{k-1}$ is upper hemicontinuous because the correspondences $E: S^{k-1} \to Z$ and $P: Z \to S^{k-1}$ are upper hemicontinuous. Then we can apply Kakutani theorem. We obtain that there exists a fixed point, that is a pair (p^*, z^*) such that $(p^*, z^*) \in \phi(p^*, z^*)$ and so $p^* \in P(z^*)$ and $z^* \in E(p^*)$. The first condition implies $pz^* \leq p^* z^*$ for every $p \in S^{k-1}$. The second one implies, by the weak Walras law, $p^* z^* \leq 0$. Therefore, we get $pz^* \leq 0$ for every $p \in S^{k-1}$. Considering the vertexes of the simplex S^{k-1}, that is points with $p_h = 1$ and $p_r = 0$ for every $r = 1,...,k$ with $r \neq h$, and this for every $h = 1,...,k$, we find out that $z_h^* \leq 0$ for every h. Therefore $z^* \in -\mathbb{R}_+^k$ and, since $z^* \in E(p^*)$, we get that there exists a $p^* \in S^{k-1}$ for which $E(p^*) \cap (-\mathbb{R}_+^k) \neq \varnothing$. □

In order to calculate the competitive equilibrium of a production economy we first determine the choices of the firms (as shown in Paragraph 5.4) and the consumers (as shown in Paragraph 3.8, with $m_i = p\omega_i + \sum_{j=1}^{m} \theta_{ij} \pi_j$, where $\pi_j = ps_j(p) = \max_{y_j \in Y_j} py_j$ for every $j = 1,...,m$) and then impose the feasibility condition $\sum_{i=1}^{n} x_i^* \leq \sum_{j=1}^{m} y_j^* + \sum_{i=1}^{n} \omega_i$.

11.9 A graphical representation of production equilibrium with two goods, one consumer and one producer

Production economies with one consumer, one producer and two goods can be represented graphically. (An economy with only one consumer resembles the case of *Robinson Crusoe* and is often called with this name). Now, this economy, as well as the pure exchange economy with two consumers and two goods (represented by Edgeworth-Pareto box diagram) is per se of a little relevance. In fact, in addition to the arguments provided in Paragraph 11.5, we assume that there is now only one individual, who is on one hand a consumer and on the other a producer, so that he sells, as consumer, a good to himself as producer and buys, as consumer, the other good from himself as producer. Moreover, he always acts as a price-taker. Also this time, the diagram will turn out to be a very useful instrument for understanding many aspects of general equilibrium of a production economy.

The examined economy is $\mathcal{E} = (\langle X, \succsim \rangle, Y, \omega)$. We note that it is necessarily $\theta = 1$. We assume $X = \mathbb{R}^2_+$, $\omega \in \mathbb{R}^2_+$ and $Y \subset \mathbb{R}^2$ non-empty, closed and, moreover, with $0 \in Y$ (possibility to take no action), $Y - \mathbb{R}^2_+ \subset Y$ (free disposal) and $Y \cap (-Y) = 0$ (irreversibility). (For these assumptions see Paragraph 5.1. Note that free disposal and irreversibility imply that there are no semipositive vectors in Y).

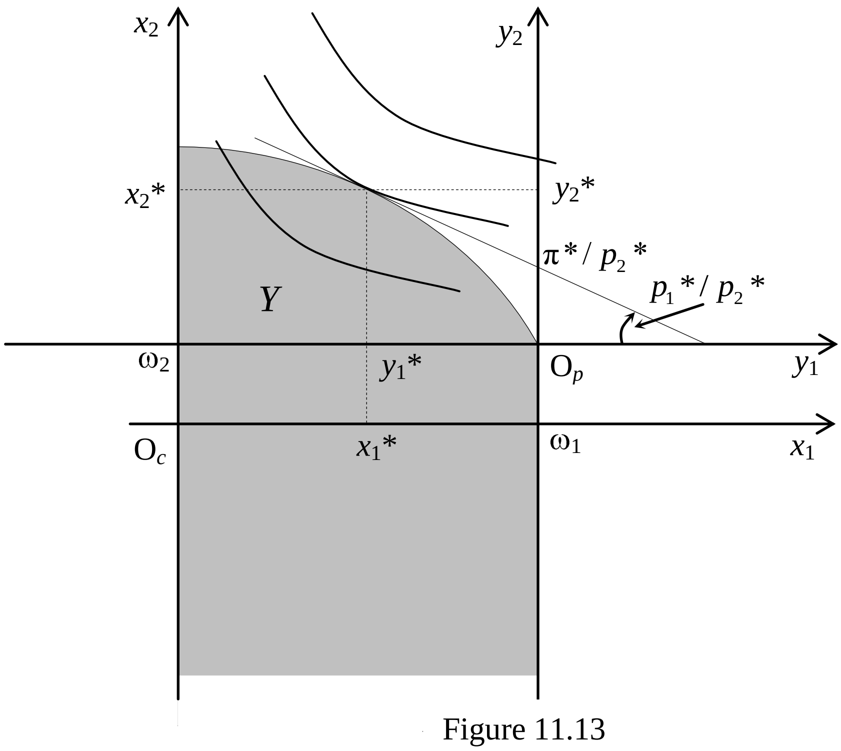

Figure 11.13

Figure 11.13 depicts the proposed diagram. There are two systems of Cartesian axes with O_c being the origin for the consumer and O_p being the origin for the producer. This point represents the endowment ω in the consumer system. With respect to the origins, the horizontal axis represents the quantity of the first good for the consumer (denoted with variable x_1) and for the producer (with a variable y_1, which is negative when, as in the figure, the first good is an input). The vertical axis indicates, analogously, the quantity of the second good. The figure shows all the data about the economy under consideration: indifference curves, which map, with respect to the origin O_c, consumer's preferences (in this figure they are regular, continuous and strongly monotone), his endowment (in the figure, the endowment is composed of a positive quantity of both goods) and, with respect to the origin O_p, the production set (in the figure it is strictly convex in the part that is relevant for the production choice, by which there are decreasing returns to scale).

The producer chooses the production level that maximizes profit (like in Figure 5.14) and the realized profit belongs to the consumer. Then, for every exchange ratio p_1/p_2, consumer's budget line, given by equation $x_1\frac{p_1}{p_2}+x_2=\omega_1\frac{p_1}{p_2}+\omega_2+\frac{\pi}{p_2}$, goes through point $(\omega_1,\omega_2+\frac{\pi}{p_2})$ and has slope equal to $-p_1/p_2$. Equilibrium occurs if the exchange ratio p_1^*/p_2^* determines a producer's choice $y^*\in\arg\max_{y\in Y} p^*y$ (with corresponding profit $\pi^* = p^*y^*$) and a consumer's choice $x^*\in\arg\max_{x\in B(p^*,y^*)} u(x)$ that satisfy the feasibility condition $x^* \le y^*+\omega$ (in our figure, where preferences are strongly monotone so there are no free goods, this condition is equivalent to $x^* = y^*+\omega$). By construction, the allocations (x, y) that satisfy $x = y+\omega$ are represented by the same point (of course x and ω are measured with respect to O_c while y with respect to O_p). In Figure 11.13 also the competitive equilibrium allocation is shown.

We note that the competitive equilibrium determines the same allocation that an individual would reach if instead of separating himself into a consumer and a producer he would choose the allocation directly. This choice solves the problem $\max_{x,y} u(x)$ subject to $x\in X$, $y\in Y$ and $x \le y+\omega$. So it is an efficient allocation.

The reader can try to represent the case in which the production set exhibits constant returns to scale (like in Figure 5.1).

If the returns to scale are first increasing and then decreasing (like in Figure 5.8) then the competitive equilibrium may not exist (like in Figure 11.14), while an efficient allocation that is a solution to the problem $\max_{x,y} u(x)$ subject to $x\in X$, $y\in Y$ and $x \le y+\omega$ exists. In Figure 11.14, there is an efficient allocation that is described by consumption x^* and production y^* (represented by the same point because this allocation is feasible).

However, with respect to the exchange ratio that is implicit in this allocation (equal to the common slope of the production set boundary in y^* and the indifference curve in x^*) the choice of the producer is inaction, that is $\hat{y} = 0$ (note, that, with this exchange ratio, the profit in y^* is negative) and the choice of the consumer is indicated by $\hat{x}$, but this allocation is not feasible (in fact, in this figure, the points $\hat{x}$ and $\hat{y}$ do not coincide). The firm chooses to produce only if the exchange ratio p_1/p_2 is sufficiently low, equal at least to the slope of the line (dashed in the figure) that determines production $\tilde{y}$. However, with this exchange ratio, the consumption choice is $\tilde{x}$ and also the allocation $\tilde{x}, \tilde{y}$ is unfeasible. With lower exchange ratios, firm's choice moves to the left, where the marginal rates of substitution of the consumer are higher (thus, to the left of $\tilde{y}$, the indifference curves are not tangent to the boundary of the production set, but intersect it), and such choices are not feasible. Therefore, there is no competitive equilibrium for the case represented in Figure 11.14.

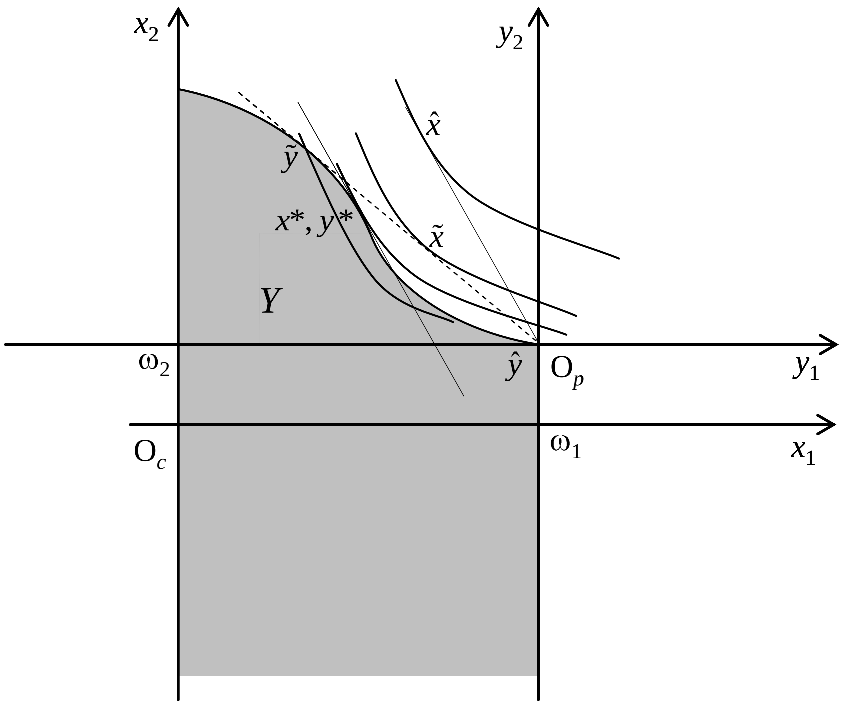

Figure 11.14

11.10 Competitive equilibrium in production economy and its efficiency

All the remarks about the pure exchange equilibrium stated in Paragraphs 11.6 and 11.7 can be extended to production economy. Naturally, we have to take into account production sets. The propositions introduced in

Paragraphs 11.6 and 11.7 for a pure exchange economy are described by the following propositions for a production economy.

Proposition 11.12 (*First welfare theorem*) If (x^*, y^*, p^*), where $x^* = (x_i^*)_{i=1}^n$ and $y^* = (y_j^*)_{j=1}^m$, is a competitive equilibrium with free disposal for an economy (without externalities) $\mathcal{E} = (\langle X_i, \succsim_i\rangle, Y_j, \omega_i, \theta_{ij}, i = 1,\ldots, n, j = 1,\ldots, m)$, then the allocation (x^*, y^*) is weakly efficient.

Proof. Consider the equivalent proposition according to which if an allocation is not weakly efficient, then it cannot belong to any competitive equilibrium. If $(\tilde{x}, \tilde{y})$ is not a weakly efficient allocation, then there exists another feasible allocation (x', y'), that is with $\Sigma_{i=1}^n x_i' \le \Sigma_{i=1}^n \omega_i + \Sigma_{j=1}^m y_j'$, such that $x_i' \succ_i \tilde{x}_i$ for every $i = 1,\ldots, n$. As a result, for every $p \in S^{k-1}$ we get $\Sigma_{i=1}^n px_i' \le \Sigma_{i=1}^n p\omega_i + \Sigma_{j=1}^m py_j'$ and, thus, $\Sigma_{i=1}^n px_i' \le \Sigma_{i=1}^n p\omega_i + \Sigma_{j=1}^m py_j^*$ since $py_j^* = \max_{y_j \in Y_j} py_j$. Therefore, there is at least one i for which $px_i' \le p\omega_i + \Sigma_{j=1}^m \theta_{ij} py_j^*$. But, if there is, for every $p \in S^{k-1}$, some i for which $px_i' \le p\omega_i + \Sigma_{j=1}^m \theta_{ij} \max_{y_j \in Y_j} py_j$ and $x_i' \succ_i \tilde{x}_i$, then $\tilde{x}_i \notin d_i(p)$ and so it is impossible that allocation $(\tilde{x}, \tilde{y})$ belongs to a competitive equilibrium. □

Proposition 11.13 If (x^*, y^*, p^*) is a competitive equilibrium (with free disposal) of an economy (without externalities) $\mathcal{E} = (\langle X_i, \succsim_i\rangle, Y_j, \omega_i, \theta_{ij}, i = 1,\ldots, n, j = 1,\ldots, m)$, and consumers have regular (that is, complete and transitive) and locally non satiated preferences, then the allocation (x^*, y^*) is strongly efficient.

Proof. Suppose that (x^*, y^*, p^*) is a competitive equilibrium but allocation (x^*, y^*) is not strongly efficient. Then there exists a feasible allocation (x, y), so with $\Sigma_{i=1}^n x_i \le \Sigma_{i=1}^n \omega_i + \Sigma_{j=1}^m y_j$, such that $x_i \succsim_i x_i^*$ for every $i = 1,\ldots,n$ and $x_s \succ_s x_s^*$ for at least one s. As a consequence, since $x_s^* \in d_s(p^*)$, it must be true for this consumer that $p^* x_s > p^*\omega_s + \Sigma_{j=1}^m \theta_{sj} p^* y_j^*$, while for all the other consumers $p^* x_i \ge p^*\omega_i + \Sigma_{j=1}^m \theta_{ij} p^* y_j^*$ (otherwise, if $p^* x_i < p^*\omega_i + \Sigma_{j=1}^m \theta_{ij} p^* y_j^*$, since the preferences are locally non satiated, there would exist a $x_i' \succ_i x_i \succsim_i x_i^*$ in the neighborhood of x_i such that $p^* x_i' < p^*\omega_i + \Sigma_{j=1}^m \theta_{ij} p^* y_j^*$, which is in contradiction with the assumed $x_i^* \in d_i(p^*)$). As a result $\Sigma_{i=1}^n p^* x_i > \Sigma_{i=1}^n p^*\omega_i + \Sigma_{j=1}^m p^* y_j^*$, that is $\Sigma_{i=1}^n p^*\omega_i + \Sigma_{j=1}^m p^* y_j^* < \Sigma_{i=1}^n p^* x_i \le \Sigma_{i=1}^n p^*\omega_i + \Sigma_{j=1}^m p^* y_j$, in contradiction to the condition $\Sigma_{j=1}^m p^* y_j^* \ge \Sigma_{j=1}^m p^* y_j$ required by the condition of profit maximization. □

Proposition 11.14 (*Second welfare theorem*) Let (x^*, y^*) be an efficient allocation of an economy (without externalities) $\mathcal{E} = (\langle X_i, \succsim_i \rangle, Y_j, \Omega, i = 1,\ldots, n, j = 1,\ldots, m)$. If the aggregate production set $Y = \Sigma_{j=1}^m Y_j$ is convex and, for every $i = 1,\ldots, n$, the consumption set X_i is convex and bounded from below, the system of preferences $\langle X_i, \succsim_i \rangle$ is regular (complete and transitive), continuous, strongly monotone[10] and convex, and x_i^* is an interior point in X_i, then there exists a vector $p^* \in S^{k-1}$ for which (x^*, y^*, p^*) is a competitive equilibrium for an economy $\mathcal{E} = (\langle X_i, \succsim_i \rangle, Y_j, \omega_i, \theta_{ij}, i = 1,\ldots, n, j = 1,\ldots, m)$ where the endowments ω_i and θ_{ij} are such that $\Sigma_{i=1}^n \omega_i = \Omega$ and $p^*(\omega_i + \Sigma_{j=1}^m \theta_{ij} y_j^*) = p^* x_i^*$ for every $i = 1,\ldots, n$.

Proof. The proof is analogous to the proof of Proposition 11.8 and we will follow a similar procedure whenever possible.

a) Let's consider, for every consumer, the sets of consumptions preferred to x_i^*, that is $P_i(x_i^*) = \{x_i \in X_i : x_i \succ_i x_i^*\}$, and their sum $P(x^*) = \Sigma_{i=1}^n P_i(x_i^*) = \{X \in \Sigma_{i=1}^n X_i : X = \Sigma_{i=1}^n x_i \text{ and } x_i \succ_i x_i^* \text{ for every } i = 1,\ldots,n\}$. All these sets are convex, since preferences are convex and the sum of convex sets is a convex set.

b) The allocation (x^*, y^*) is efficient, so not only $X^* \notin P(x^*)$, where $X^* = \Sigma_{i=1}^n x_i^*$, but also the sets $P(x^*)$ and $G = Y+\{\Omega\}$ are disjoint. Therefore we can apply the separating hyperplane theorem,[11] according to which there

[10] The assumption of strongly monotone preferences could be substituted with a weaker assumption that requires the preferences to be locally nonsatiated. In such a case, the proof has to be slightly modified.

[11] There are different formalizations of the separating hyperplane theorem. The one relevant for the examined proposition says that if $P,G \subset \mathbb{R}^k$ is a pair of convex disjoint sets, then there exists a vector $a \neq 0$ such that $ax \geq ag$ for every pair x,g with $x \in P$ and $g \in G$. In Figure 11.15 we depict a case in which P is a convex and open set and G is a closed set.

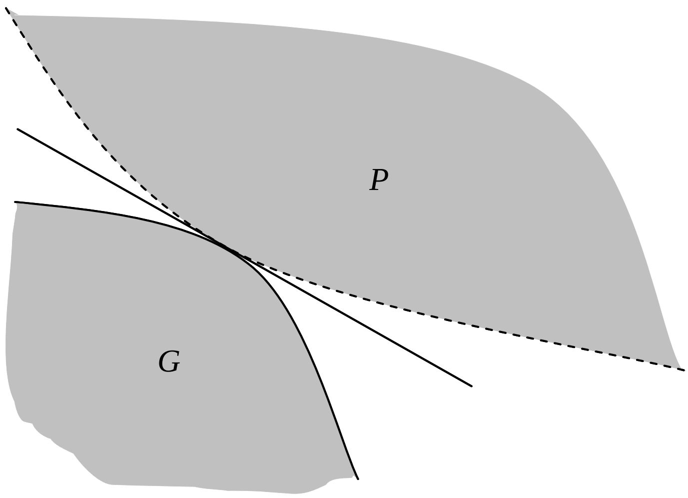

Figure 11.15

exists a vector $a \neq 0$ and a scalar r such that $aX \geq r \geq ag$ for every $X \in P(x^*)$ and every $g \in G$.

c) For every $i = 1,...,n$, since x_i^* is a point in the interior of X_i and the preferences are continuous and monotone, there is a $x_i \succ_i x_i^*$ in each ball with center x_i^*. Therefore, for any allocation $(x_i)_{i=1}^n$ composed of those points, having $\Sigma_{i=1}^n x_i \in P(x^*)$, we get $a\Sigma_{i=1}^n x_i \geq r$. Considering balls with smaller and smaller radius, approaching zero, we obtain, by continuity, that $a\Sigma_{i=1}^n x_i^* \geq r$. On the other hand, since preferences are strongly monotone, an efficient allocation (x^*, y^*) satisfies the condition $\Sigma_{i=1}^n x_i^* = \Omega + \Sigma_{j=1}^m y_j^*$. Then, since $\Omega + \Sigma_{j=1}^m y_j^* \in G$, it follows that $r \geq a(\Omega + \Sigma_{j=1}^m y_j^*) = a\Sigma_{i=1}^n x_i^*$. As a consequence $aX^* = a\Sigma_{i=1}^n x_i^* = r$.

d) Let's consider the allocation $(x_i^* + \frac{1}{n} e_h)_{i=1}^n$, where e_h is a vector with all components equal to zero except h-th which is equal to 1. Since the preferences are strongly monotone we get $(x_i^* + \frac{1}{n} e_h) \succ_i x_i^*$ for every $i = 1,...,n$. Therefore, since $X^* + e_h = \Sigma_{i=1}^n (x_i^* + \frac{1}{n} e_h)$, then $X^* + e_h \in P(x^*)$ for every $h = 1,...,k$. Then, keeping in mind that $aX^* = r$ and applying the inequality from the separating hyperplane theorem we get $a(X^* + e_h) \geq r = aX^*$, that is $a_h \geq 0$ for every $h = 1,...,k$. At this point, we define $p^* = a\dfrac{1}{\Sigma_{h=1}^k a_h}$ and obtain that $p^* \in S^{k-1}$ and $p^* X \geq r\dfrac{1}{\Sigma_{h=1}^k a_h} \geq p^* g$ for every $X \in P(x^*)$ and every $g \in G$, with $r\dfrac{1}{\Sigma_{h=1}^k a_h} = p^* X^*$.

e) The relationships $p^* X^* \geq p^* g$ for every $g \in G$, where $G = Y + \{\Omega\}$, and $X^* = \Sigma_{i=1}^n x_i^* = \Omega + \Sigma_{j=1}^m y_j^*$ imply that $p^* \Sigma_{j=1}^m y_j^* \geq p^* \Sigma_{j=1}^m y_j$ for every $\Sigma_{j=1}^m y_j \in Y$. It means that $p^* \Sigma_{j=1}^m y_j^* = \max_{\Sigma_{j=1}^m y_j \in Y} p^* \Sigma_{j=1}^m y_j$, that is, keeping in mind Proposition 5.15, that $p^* y_j^* = \max_{y_j \in Y_j} p^* y_j$, so $y_j^* \in s_j(p^*)$ for every $j = 1,...,m$.

f) We now prove, for every $i = 1,...,n$, that if $x_i \succ_i x_i^*$ then $p^* x_i \geq p^* x_i^*$. If $x_i \succ_i x_i^*$, since the preferences are convex we obtain that $x_i' \succ_i x_i^*$ for every $x_i' = \lambda x_i + (1-\lambda) x_i^*$ with $\lambda \in (0,1]$. Let's take x_i' sufficiently close to x_i^* so to have a point in the interior of X_i (by assumption, x_i^* is a point in the interior of X_i). Then, since preferences are continuous, there exists a $x_i'' << x_i'$ around x_i' in X_i such that $x_i'' \succ_i x_i^*$. Let's take under consideration the consumption allocation x'', where x_i'' is already introduced and $x_b'' = x_b^* + \frac{1}{n-1}(x_i' - x_i'')$ for every $b \neq i$ and $b = 1,...,n$. Since the preferences are monotone we get $x_i'' \succ_i x_i^*$ for every

$i=1,...,n$. Then we obtain $X''=\Sigma_{i=1}^{n} x_i'' \in P(x^*)$ and so, by separating hyperplane theorem, $p^*X'' \geq r = p^*X^*$, that is $p^*\Sigma_{i=1}^{n} x_i'' \geq p^*\Sigma_{i=1}^{n} x_i^*$. Keeping in mind the definition of allocation x'' we obtain

$$p^*x_i''+p^*\Sigma_{b=1,b\neq i}^{n}(x_b^*+\tfrac{1}{n-1}(x_i'-x_i''))\geq p^*\Sigma_{b=1}^{n} x_b^*$$

and from this, as a result, $p^*x_i' \geq p^*x_i^*$. Thus, keeping in mind that $x_i'=\lambda x_i+(1-\lambda)x_i^*$, we have $\lambda p^*x_i \geq \lambda p^*x_i^*$, that is $p^*x_i \geq p^*x_i^*$ since $\lambda \in (0,1]$.

g) We can strengthen the preceding relationship by proving that if $x_i \succ_i x_i^*$ then $p^*x_i > p^*x_i^*$ (not simply $p^*x_i \geq p^*x_i^*$). In fact, as already noted, if $x_i \succ_i x_i^*$, since the preferences are convex and continuous and x_i^* is a point in the interior of X_i, there exists a $x_i'' << x_i'$, where $x_i'=\lambda x_i+(1-\lambda)x_i^*$ with $\lambda \in (0,1]$, such that $x_i'' \succ_i x_i^*$. Then applying the relationship established in the preceding step, we get $p^*x_i'' \geq p^*x_i^*$ and so, $p^*x_i' > p^*x_i'' \geq p^*x_i^*$. Having $x_i'=\lambda x_i+(1-\lambda)x_i^*$ with $\lambda \in (0,1]$, the inequality $p^*x_i' > p^*x_i^*$ is equivalent to $p^*x_i > p^*x_i^*$.

h) From the preceding step, we get that if $x_i \succ_i x_i^*$ then $p^*x_i > p^*x_i^*$ for every $i=1,...,n$. This implies that if $p^*x_i \leq p^*x_i^*$ then $x_i \precsim_i x_i^*$ and so it is $x_i^* \succsim_i x_i$ for every $x_i \in \ x_i \in X_i : p^*x_i \leq p^*x_i^*$. Therefore, since $p^*x_i^* = p^*(\omega_i + \Sigma_{j=1}^{m} \theta_{ij} y_j^*)$ by the definition of the endowments ω_i and θ_{ij}, we get $x_i^* \in d_i(p^*, p^*(\omega_i + \Sigma_{j=1}^{m} \theta_{ij} y_j^*))$ for every $i=1,...,n$.

i) Results obtained in steps *e*) and *h*), that is $y_j^* \in s_j(p^*)$ for every $j=1,...,m$ and $x_i^* \in d_i(p^*, p^*(\omega_i + \Sigma_{j=1}^{m} \theta_{ij} y_j^*))$ for every $i=1,...,n$, prove, if we keep in mind that an efficient allocation satisfies the feasibility condition, that (x^*, y^*, p^*) is a competitive equilibrium. (Finally, note that this result and the assumption that the preferences are strongly monotone imply also that $p^* >> 0$). □

The following propositions describe the relationship between the allocations that belong to a competitive equilibrium and those that maximize social welfare.

Proposition 11.15 Let's consider an economy (without externalities) $(\langle X_i, \succsim_i \rangle, Y_j, \Omega, i=1,\ldots,n, j=1,\ldots,m)$. Let (x^*, y^*), where $x^*=(x_i^*)_{i=1}^{n}$ and $y^*=(y_j^*)_{j=1}^{m}$, be an allocation that maximizes a social welfare function $W(u_1(x_1),...,u_n(x_n))$ over the set of feasible allocations $C_{FD}=\{((x_i)_{i=1}^{n},(y_j)_{i=1}^{m}) : x_i \in X_i, y_j \in Y_j, \sum_{i=1}^{n} x_i \leq \sum_{j=1}^{m} y_j + \Omega\}$. If the aggregate production set $Y = \Sigma_{j=1}^{m} Y_j$ is convex and, for every $i=1,...,n$, the consumption set X_i is convex and bounded from below, the system of preferences $\langle X_i, \succsim_i\rangle$ is regular (that is complete and transitive), continuous, strongly monotone and convex and x_i^* is a point in the interior of X_i, then there exists a vector $p^* \in S^{k-1}$ for which (x^*, y^*, p^*) is a competitive equilibrium with free disposal for an economy $\mathcal{E} = (\langle X_i, \succsim_i\rangle, Y_j, \omega_i, \theta_{ij}, i = 1,\ldots, n, j = 1,\ldots, m)$, where the endowments ω_i and θ_{ij} are such that $\Sigma_{i=1}^n \omega_i = \Omega$ and $p^*(\omega_i + \Sigma_{j=1}^m \theta_{ij} y_j^*) = p^* x_i^*$ for every $i = 1,\ldots,n$.

Proof. The proof follows directly from Propositions 8.9 and 11.14. □

Proposition 11.16 If (x^*, y^*, p^*) is a competitive equilibrium with free disposal for an economy (without externalities) $\mathcal{E} = (\langle X_i, \succsim_i\rangle, Y_j, \omega_i, \theta_{ij}, i = 1,\ldots, n, j = 1,\ldots, m)$, then the allocation (x^*, y^*) maximizes at least one social welfare function. In particular, if the preferences of the consumers can be represented with concave and monotone utility functions and the aggregate production set is convex, this allocation maximizes the social welfare function $\Sigma_{i=1}^n \frac{1}{\lambda_i} u_i(x_i)$ over the set $C_{FD} = \{((x_i)_{i=1}^n, (y_j)_{i=1}^m) : x_i \in X_i$, $y_j \in Y_j$, $\Sigma_{i=1}^n x_i \le \Sigma_{j=1}^m y_j + \Omega\}$, where λ_i is the marginal indirect utility of wealth for the i-th consumer, that is $\lambda_i = \mathrm{D}_{m_i} u_i^*(p^*, p^*\omega_i + \Sigma_{j=1}^m \theta_{ij} p^* y_j^*)$, for every $i = 1,\ldots,n$.

Proof. The proof is analogous to the one in Proposition 11.10. The first part of the proposition is a direct consequence of Propositions 8.10 and 11.12. For the second part we can show that the maximization of the proposed social welfare function leads to the competitive equilibrium allocation. In fact, introducing for the problem $\max_{(x,y)\in C_{FD}} \Sigma_{i=1}^n \frac{1}{\lambda_i} u_i(x_i)$ the Lagrangian function

$\mathrm{L}(x_1,\ldots, x_n, y_1,\ldots, y_m, \mu_1,\ldots, \mu_k, \nu_1,\ldots, \nu_m) =$

$$\Sigma_{i=1}^n \frac{1}{\lambda_i} u_i(x_i) + \Sigma_{h=1}^k \mu_h (\Sigma_{i=1}^n \omega_{ih} + \Sigma_{j=1}^m y_{jh} - \Sigma_{i=1}^n x_{ih}) - \Sigma_{j=1}^m \nu_j F_j(y_j),$$

we get the following first order conditions

$\frac{1}{\lambda_i} \mathrm{D}_{x_{ih}} u_i(x_i) = \mu_h$ for every $i = 1,\ldots, n$ and every $h = 1,\ldots, k$,

$\mu_h = \nu_j \mathrm{D}_{y_{jh}} F_j(y_j)$ for every $j = 1,\ldots, m$ and every $h = 1,\ldots, k$,

$\Sigma_{i=1}^n \omega_{ih} + \Sigma_{j=1}^m y_{jh} - \Sigma_{i=1}^n x_{ih} = 0$ for every $h = 1,\ldots, k$,

$F_j(y_j) = 0$ for every $j = 1,\ldots, m$.

The solution (x, y, μ, ν) of these conditions is coherent with competitive equilibrium (x^*, y^*, p^*), since they coincide with the first order conditions for competitive equilibrium having $\lambda_i = \mathrm{D}_{m_i} u_i^*(p^*, p^*\omega_i + \Sigma_{j=1}^m \theta_{ij} p^* y_j^*)$ for every $i = 1,\ldots,n$. In fact, the competitive equilibrium conditions, which are composed of the relationships $\mathrm{D}_{x_{ih}} u_i(x_i) = \lambda_i p_h$, $p_h = \nu_j \mathrm{D}_{y_{jh}} F_j(y_j)$ and

$\Sigma_{i=1}^{n}\omega_{ih} + \Sigma_{j=1}^{m}y_{jh} - \Sigma_{i=1}^{n}x_{ih} = 0$ for every $i = 1,...,n$, $j = 1,...,m$ and $h = 1,...,k$, and also of budget constraints $px_i = p\omega_i + \Sigma_{j=1}^{m}\theta_{ij}py_j$ for every $i = 1,...,n$, require $\lambda_i = \mathrm{D}_{m_i}u_i*(p*, p*\omega_i + \Sigma_{j=1}^{m}\theta_{ij}p*y_j*) = \mathrm{D}_{m_i}u_i*(p*, p*x_i*)$ for every $i = 1,...,n$, so that budget constraints are implicitly assumed also by the problem $\max_{(x,y)\in C_{FD}} \Sigma_{i=1}^{n}\frac{1}{\lambda_i}u_i(x_i)$. Finally, concavity of utility functions and convexity of the aggregate consumption sets, on one hand, imply that second order conditions are satisfied and, on the other hand, that the first order conditions determine the global social welfare maximum. □

11.11 Equilibrium with free entry and the non-substitution theorem

In this paragraph the main goal is to determine the conditions that yield the prices of products independent of the preferences and the endowments, that is dependent only on the production sets. In this way we will be able to establish the conditions that yield on one hand that the classical theory of prices holds (according to which prices are not only equal to production costs but are uniquely determined by them, which are in turn determined only by technology) and, on the other hand, that would put forward the Leontief input-output model that uses constant production coefficients (as shown in footnote 15 in Chapter 5). This goal can be achieved if we suppose, together with other assumptions, that the industry production sets exhibit constant returns to scale. This property is satisfied if we allow for free entry. In fact in § 5.8 (with Definition 5.6) we introduced the *industry production set with free entry* and proved (in Proposition 5.16) that it exhibits constant returns to scale.

For example, in an economy with an industry that has only one input and only one output and the production set as depicted in Figure 5.1, the competitive equilibrium (if it exists and the examined industry is active) requires that the exchange ratio between the two goods (input and output of the industry) is determined only by the transformation function (which represents the relevant part of the boundary of the production set and is of the type $y_2+ay_1 = 0$): we have $p_1*/p_2* = a$ independently of the demand and supply functions by consumers.

We, therefore, examine the properties of the competitive equilibrium in an economy without joint production (that is with only one good produced by each industry) that is characterized by free entry for every industry. For this economy, since production exhibits constant returns to scale and results in zero profits, the prices of the goods produced (in positive quantity) are equal to their average production costs. The following Proposition 11.17 introduces conditions that establish that these equalities are sufficient for price determination.

We thus examine the competitive equilibrium with free disposal (x^*, y^*, p^*) of a production economy with free entry $\mathcal{E} = (\langle X_i, \succsim_i\rangle, \hat{Y}_j, \omega_i, \theta_{ij}, i = 1,\ldots, n, j = 1,\ldots, m)$, where every industry production set $\hat{Y}_j \subset \mathbb{R}^k$ has only one output (and every set has a different output) and exhibits constant returns to scale. In this equilibrium, the revenue of every industry is equal to its (minimal) production cost that is proportional (having constant returns to scale) to the quantity produced. To be precise, listing the goods in a way that the first m goods are the products, we have $y_j^* = (q_j^*, \xi_j^*)$, with $q_j^* \in \mathbb{R}_{++}$ and $\xi_j^* \in \mathbb{R}_{+}^{k-1}$, and $p_j^* q_j^* = c_j^*(p_{\xi_j}^*, q_j^*) = AC_j^*(p_{\xi_j}^*)\, q_j^*$ for every $j = 1,\ldots,m$. Then $p_j^* = AC_j^*(p_{\xi_j}^*)$ if $q_j^* > 0$, where p_j^* is the price of the j-th industry output, $p_{\xi_j}^*$ is a vector of prices of its input and AC_j^* is the average cost, which is a strongly monotone function of the prices of inputs used in positive amount.

Proposition 11.17 (*The Non-substitution Theorem*) If there is no joint production, if the production sets have constant returns to scale, if there is only one input which is *primary* (i.e. a non-produced good) and if the primary good is necessary in every production (that is, every industry uses a positive quantity of the primary good), then the prices of the products are determined only by their costs. That is, listing the goods in a way that the first m goods are the products and the $(m+1)$-th good is the primary good, then the system of equations $p_j = AC_j^*(p_1,\ldots,p_m,p_{m+1})$, with $j = 1,\ldots,m$, admits a unique solution.

Proof. First of all, if we choose the primary good as numeraire, that is $p_{m+1} = 1$, since it is used in every production, then all the average production costs are positive, so $p_j > 0$ for $j = 1,\ldots,m$. Suppose, now, that the system $p_j = AC_j^*(p_1,\ldots,p_m,p_{m+1})$, $j = 1,\ldots,m$, admits two solutions: the competitive equilibrium solution p^* and some other solution p'. Consider the number $\alpha = \max_j \left\{ \frac{p_j'}{p_j^*} \right\}$. If $\alpha > 1$, then there is a good h, with $h \in \{1,\ldots, m\}$, for which $p_h' = \alpha p_h^* > p_h^*$, while for the other products we have $p_j' \le \alpha p_j^*$. If $\alpha \le 1$, then, since $p^* \ne p'$, there is a good h for which we have $p_h' = \beta p_h^* < p_h^*$, with $\beta = \min_j \left\{ \frac{p_j'}{p_j^*} \right\} < 1$, and $p_j' \ge \beta p_j^*$ for the other products. The following reasoning, done for the case $\alpha > 1$, applies to both cases, with some modification. Keeping in mind that it must be $p_h^* = AC_h^*(p_1^*,\ldots,p_m^*,p_{m+1})$ and $p_h' = AC_h^*(p_1',\ldots,p_m',p_{m+1})$, that the cost function is homogenous of degree 1, by which $\alpha p_h^* = AC_h^*(\alpha p_1^*,\ldots,\alpha p_m^*,\alpha p_{m+1})$, and that it is strongly monotone with respect to the prices of employed inputs, by which

$AC_h*(\alpha p_1*,...,\alpha p_m*,\alpha p_{m+1}) > AC_h*(p_1',...,p_m',p_{m+1})$ since $\alpha p_j* \geq p_j'$, for $j=1,...,m$, and $\alpha p_{m+1} > p_{m+1}$, we get the relationships

$$p_h' = \alpha p_h* = \alpha AC_h*(p_1*,...,p_m*,p_{m+1}) = AC_h*(\alpha p_1*,...,\alpha p_m*,\alpha p_{m+1}) >$$
$$> AC_h*(p_1',...,p_m',p_{m+1}) = p_h',$$

that reveal a contradiction. (If $\alpha \leq 1$, then, having $\beta < 1$, we get

$$p_h' = \beta p_h* = \beta AC_h*(p_1*,...,p_m*,p_{m+1}) = AC_h*(\beta p_1*,...,\beta p_m*,\beta p_{m+1}) <$$
$$< AC_h*(p_1',...,p_m',p_{m+1}) = p_h').$$

As a consequence, there cannot be two solutions for the system of equations $p_j = AC_j*(p_1,...,p_m,p_{m+1})$, with $j=1,...,m$, which therefore has only one solution. □

The preceding proposition requires very strong assumptions, among them that there is only one input which is not produced. Usually, this input represents labor. This assumption requires that in the economy there is only one type of labor and that natural resources are not used. Nevertheless, the result that the prices of products depend only on the production costs and so are independent of the endowments and the consumption preferences is relevant also for the following reason.

In the case under consideration, basing on the Shephard's lemma (Proposition 5.22), we find out that production coefficients a_{jh} (that denote, for every $j=1,...,m$ and $h=1,...,m+1$, the quantity of input h necessary to produce one unit of output *j*) are independent of the consumers demand, too. In fact, Shephard's lemma requires that $x_{jh} = \dfrac{\partial c_j*(p_1,...,p_m,p_{m+1},q_j)}{\partial p_h}$, for every $j=1,...,m$ and $h=1,...,m+1$. So having constant returns to scale we get $a_{jh} = \dfrac{x_{jh}}{q_j} = \dfrac{\partial AC_j*(p_1,...,p_m,p_{m+1})}{\partial p_h}$ for $j=1,...,m$ and $h=1,...,m+1$. Since the average cost function is independent of the consumers choices, also the production coefficients a_{jh} do not depend on them. Then, also if the (constant returns to scale) technology is not such that the coefficients are constant (that is, with inputs that are perfect complements), the choice of the production coefficients is independent of the demand for products. In other words, change in the demand does not generate substitutions of inputs, even if they are permitted by the technology.

11.12 General competitive equilibrium without free disposal

In Paragraph 11.3 we introduced competitive equilibrium of pure exchange economy and we distinguished between economy with and without free disposal. Then, we examined equilibrium for the first case. We will now examine the equilibrium for the second case. In what follows we analyze only the pure exchange economy equilibrium. The extension to the production economy is easy and does not give additional insight on the top of the observations presented for pure exchange economy.

According to the Definition 11.4, the competitive equilibrium of a pure exchange economy $\mathcal{E} = (\langle X_i, \succsim_i\rangle, \omega_i, i = 1,\ldots, n)$ *without free disposal* is a vector of prices $p^* \in \mathbb{R}^k$ and an allocation $x^* = (x_1^*,..., x_n^*)$ such that $x_i^* \in \bar{d}_i(p^*)$, where $\bar{d}_i(p) = \{x_i \in \bar{B}_i(p) : x_i \succsim_i x_i' \text{ for every } x_i' \in \bar{B}_i(p)\}$ and $\bar{B}_i(p) = \{x_i \in X_i : px_i = p\omega_i\}$, for every $i = 1,..., n$, and $\Sigma_{i=1}^n x_i^* = \Sigma_{i=1}^n \omega_i$. (*With free disposal*, we require $x_i^* \in d_i(p^*)$, where $d_i(p) = \{x_i \in B_i(p) : x_i \succsim x_i' \text{ for every } x_i' \in B_i(p)\}$ and $B_i(p) = \{x_i \in X_i : px_i \le p\omega_i\}$, for every $i = 1,..., n$, and $\Sigma_{i=1}^n x_i^* \le \Sigma_{i=1}^n \omega_i$).[12]

[12] Usually in economic literature (for example Debreu, 1982), equilibrium without free disposal is defined by conditions $\Sigma_{i=1}^n x_i^* = \Sigma_{i=1}^n \omega_i$ and $x_i^* \in d_i(p^*)$ for every $i = 1,\ldots, n$, so in an intermediate way with respect to the definitions of "without free disposal" and "with free disposal" used in this text. In what follows, we illustrate some characteristics of this intermediate equilibrium, called also "strong equilibrium" in contraposition to the one in this text which is called "weak equilibrium" because every equilibrium (x^*, p^*) that satisfies the conditions for "strong" equilibrium, that is $\Sigma_{i=1}^n x_i^* = \Sigma_{i=1}^n \omega_i$ and $x_i^* \in d_i(p^*)$ for every $i = 1,\ldots, n$, satisfies also the conditions for "weak" equilibrium, that is $\Sigma_{i=1}^n x_i^* = \Sigma_{i=1}^n \omega_i$ and $x_i^* \in \bar{d}_i(p^*)$ for every $i = 1,\ldots, n$. In fact, the conditions $\Sigma_{i=1}^n x_i^* = \Sigma_{i=1}^n \omega_i$ and $p^*x_i^* \le p^*\omega_i$ for every $i = 1,\ldots, n$ imply $p^*x_i^* = p^*\omega_i$ for every $i = 1,\ldots, n$, that is $x_i^* \in \bar{B}_i(p^*)$ and so, $x_i^* \in \bar{d}_i(p^*)$. The inverse property does not hold as we will see with some examples (Figures 11.16 and 11.17). The reasons for which we may prefer "weak" equilibrium (that is with budget constraints $\bar{B}_i(p) \subseteq B_i(p^*)$ instead of $B_i(p)$) for the economy without free disposal are based on three convergent observations.

a) The first observation results from the payments generated by the exchange. These can be performed in a barter system, with the use of money or with the use of a credit system. If we use barter, the inequality $px_i < p\omega_i$ requires that some quantity of a good is eliminated, but this possibility is excluded without free disposal. If we use money, we have a situation equivalent to the following: the agents deposit their endowments in a common warehouse in exchange for money at given prices. They use this money to get the desired goods from the same warehouse at the same prices. If $px_i \le p\omega_i$ for every $i = 1,\ldots, n$ and $px_i < p\omega_i$ for some i, then some quantity of at least one good would remain in the warehouse, that is there would be free disposal. If, finally, the payments are credited, there is for every agent an account in which all the sales are credited and purchases are debited. Therefore, every exchange means an entry on credit for the agent that sells and on debit for the agent that buys. The sum of entries must be equal to zero. Then, the account of an agent cannot be in credit, just like we have that if $px_i < p\omega_i$ for some i, without other agent account being in debit, that is $px_i > p\omega_i$ for some other i, possibility that is excluded by the budget constraint.

b) The second observation results from taking into account that exchange ratios, and not accounting prices, are relevant in the budget constraint. Also with prices not necessarily nonnegative (negative prices can occur when free disposal is not assumed), the budget constraint has not to be modified if we put tp in place of p, with $t \ne 0$. It must be that $B_i(-p) = B_i(p)$, but this condition is not satisfied if the budget constraint is an inequality.

We need weaker assumptions to prove the existence of the competitive equilibrium in the case without free disposal than in the case with free disposal. The conditions from Proposition 11.11, that require the set X_i to be non-empty, compact and convex, the system of preferences $\langle X_i, \succsim_i\rangle$ to be regular, continuous, convex and monotone and ω_i to be a point in the interior of X_i for every $i = 1,...,n$, can weaken the conditions on preferences. It is sufficient to assume that the system of preferences $\langle X_i, \succsim_i\rangle$ is regular, continuous and weakly convex. In other terms the indifference curves may be thick and consumption of global satiation (that is, such that there is no other consumption in X_i that is preferred to it) may belong to the interior of the consumption set. The Proposition 11.18 introduces the continuity conditions that will be needed in Proposition 11.19 to prove competitive equilibrium.

Proposition 11.18 If the consumption set X_i is non-empty, compact and convex, the system of preferences $\langle X_i, \succsim_i\rangle$ is regular, continuous and weakly convex and the endowment ω_i is a point in the interior of X_i, then the budget set $\overline{B}_i(p) = \{x_i \in X_i : px_i = p\omega_i\}$ and the demand set $\overline{d}_i(p) = \ x_i \in \overline{B}_i(p) : x_i \succsim_i x_i' \text{ for every } x_i' \in \overline{B}_i(p)$ are non-empty, compact and convex for every $p \in \mathbb{R}^k$, the budget correspondence $\overline{B}_i : \mathbb{R}^k \setminus \{0\} \to X_i$ is continuous and homogenous of degree zero and the demand correspondence $\overline{d}_i : \mathbb{R}^k \setminus \{0\} \to X_i$ is upper hemicontinuous and homogenous of degree zero.

Proof. It is immediate to show that the sets $\overline{B}_i(p)$ and $\overline{d}_i(p)$ are non-empty, compact and convex and the budget and demand correspondences are homogenous of degree zero (that is $\overline{B}_i(tp) = \overline{B}_i(p)$ and $\overline{d}_i(tp) = \overline{d}_i(p)$ for every $t \neq 0$). The continuity of the budget correspondence can be proved by showing that it is both upper hemicontinuous (that is the sequences $p^q \to p^o$ and $x_i^q \to x_i^o$, with $x_i^q \in \overline{B}_i(p^q)$, imply $x_i^o \in \overline{B}_i(p^o)$) and lower hemicontinuous (that is, if $p^q \to p^o$ and $x_i^o \in \overline{B}_i(p^o)$, then there exists a sequence (x_i^q) such that $x_i^q \in \overline{B}_i(p^q)$ and $x_i^q \to x_i^o$). It is upper hemicontinuous because the set $\ (p, x_i) \in P \times X_i : x_i \in \overline{B}_i(p)$ is closed if $P \subset \mathbb{R}^k \setminus \{0\}$ is a closed set. It is lower hemicontinuous because of the following reasoning. Since ω_i is a point in the interior of X_i, there is a pair

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c) As already noted, the conditions $\Sigma_{i=1}^n x_i^* = \Sigma_{i=1}^n \omega_i$ and $p^* x_i^* \leq p^* \omega_i$ for every $i = 1,\ldots, n$ imply $p^* x_i^* = p^* \omega_i$ for every $i = 1,\ldots, n$. Taking into account that the agents know the equilibrium conditions and that no choice such that $px_i < p\omega_i$ is feasible, we deduce that the agents bound their choices to the bundles for which $px_i = p\omega_i$. Therefore, they limit their choice to the points in $\overline{B}_i(p)$ also if their budget constraint is $B_i(p)$. As a result, we get the demand functions $\overline{d}_i(p^*)$, exactly those generated by the budget constraints $\overline{B}_i(p)$.

$x_i^a, x_i^b \in X_i$ with $p^o x_i^a < p^o \omega_i < p^o x_i^b$. Then, for every sequence (p^q) with $p^q \to p^o$ and $x_i^o \in \overline{B}_i(p^o)$, let's introduce two sequences (x_i^{aq}) and (x_i^{bq}) where $x_i^{aq} = t^{aq} x_i^o + (1-t^{aq}) x_i^a$ and $x_i^{bq} = t^{bq} x_i^o + (1-t^{bq}) x_i^a$, with $t^{aq} = \dfrac{p^q(\omega_i - x_i^a)}{p^q(x_i^o - x_i^a)}$ and $t^{bq} = \dfrac{p^q(\omega_i - x_i^b)}{p^q(x_i^o - x_i^b)}$, so that $p^q x_i^{aq} = p^q x_i^{bq} = p^q \omega_i$. We find that $t^{aq}, t^{bq} > 0$ for sufficiently large *q*, because $p^q x_i^a \to p^o x_i^a$, $p^q x_i^b \to p^o x_i^b$ and $p^q x_i^o \to p^o x_i^o = p^o \omega_i$, so that $t^{aq}, t^{bq} \to 1$. Moreover, always for *q* sufficiently large, we find that if $t^{aq} > 1$ then $t^{bq} < 1$ and viceversa, because $t^{aq} > 1$ only if $p^q \omega_i > p^q x_i^o$, which implies $t^{bq} < 1$. Consequently, for sufficiently large *q*, the sequence (x_i^q), where $x_i^q = x_i^{aq}$ if $t^{aq} \le 1$ and $x_i^q = x^{bq}$ if $t^{aq} > 1$, is composed of vectors $x_i^q \in \overline{B}_i(p^q)$ because $p^q x_i^q = p^q \omega_i$, the set X_i is convex and $x_i^q = t^{aq} x_i^o + (1-t^{aq}) x_i^a$ with $0 < t^{aq} \le 1$ or $x_i^q = t^{bq} x_i^o + (1-t^{bq}) x_i^a$ with $0 < t^{bq} < 1$. Therefore, $x_i^q \to x_i^o$ because both $t^{aq} \to 1$ and $t^{bq} \to 1$ for $p^q \to p^o$. Finally, the correspondence $\overline{d}_i : \mathbb{R}^k \setminus \{0\} \to X_i$ is upper hemicontinuous because $p^q \to p^o$ and $x_i^q \to x_i^o$ with $x_i^q \in \overline{d}_i(p^q)$ imply $x_i^o \in \overline{d}_i(p^o)$ for the following reasons. On one hand, since $x_i^q \in \overline{d}_i(p^q) \subset \overline{B}_i(p^q)$ we get $x_i^o \in \overline{B}_i(p^o)$ because the correspondence $\overline{B}_i : \mathbb{R}^k \setminus \{0\} \to X_i$ is upper hemicontinuous. On the other hand, for every point $z \in \overline{B}_i(p^o)$, since the correspondence $\overline{B}_i : \mathbb{R}^k \setminus \{0\} \to X_i$ is also lower hemicontinuous, there is a sequence (z^q) such that $z^q \in \overline{B}_i(p^q)$ and $z^q \to z$. Then, since $x_i^q \succsim_i z^q$ because $x_i^q \in \overline{d}_i(p^q)$ and since preferences are continuous, we get $x_i^o \succsim_i z$. Since this relationship holds for every $z \in \overline{B}_i(p^o)$, we obtain $x_i^o \in \overline{d}_i(p^o)$. □

Homogeneity of the budget correspondences and of the demand correspondences allows us to standardize the prices, keeping in mind that they can also be negative since we do not assume free disposal. Therefore, we cannot consider the simplex as the set of prices. Let's standardize, the prices by putting the norm of the vector of prices equal to 1. Therefore, the set of prices is the sphere with the radius equal to 1, that is

$$S = \{ p \in \mathbb{R}^k : \|p\| = 1 \}$$

We note that the set *S* is not convex and so we cannot use Kakutani theorem.[13] We can, nevertheless, use the following theorem.

[13] Moreover, set *S* excludes the vector $p = 0$. This case, however, is irrelevant from the economic point of view. In fact, if $p = 0$, then $\overline{B}_i(0) = X_i$ for every $i = 1,\ldots, n$ and every agent can choose his satiation consumption. Then the equilibrium would exist only if the

Theorem:[14] Let $S = \ p \in \mathbb{R}^k : \|p\| = 1\ $ and let $Z \subset \mathbb{R}^k$ be a compact set. If $\phi : S \to Z$ is an upper hemicontinuous correspondence with set $\phi(p)$ non-empty and convex for every $p \in S$, then *at least* one of the following three alternatives is true:

a) there exists a $p^* \in S$ such that $0 \in \phi(p^*)$;

b) there exists a pair (p^*, z^*) with $p^* \in S$ and $z^* \in \phi(p^*)$ such that $z^* \neq 0$ and $\dfrac{z^*}{\|z^*\|} = p^*$;

c) there exists a triple $(p^*, z^*, \hat{z})$ with $p^* \in S$, $z^* \in \phi(p^*)$ and $\hat{z} \in \phi(-p^*)$ such that $z^*, \hat{z} \neq 0$ and $\dfrac{\hat{z}}{\|\hat{z}\|} = -\dfrac{z^*}{\|z^*\|}$.

Let's take under consideration the aggregate excess demand correspondence $\bar{E} : S \to Z$, where $\bar{E}(p) = \sum_{i=1}^{n} (\bar{d}_i(p) - \omega_i)$. The set $Z = \sum_{i=1}^{n} X_i - \{\sum_{i=1}^{n} \omega_i\}$ is a non-empty and compact subset of $\mathbb{R}^k$ if the consumption sets X_i, for $i = 1, ..., n$, are non-empty and compact. The correspondence $\bar{E} : S \to Z$ is, under assumptions from Proposition 11.18, upper hemicontinuous, with sets $E(p)$ being non-empty, compact and convex for every $p \in S$. Moreover, since $\bar{d}_i(p) \subseteq \bar{B}_i(p)$, Walras law holds, that is we have $p\bar{E}(p) = 0$ for every $p \in S$ (with $p\bar{E}(p) = 0$ we denote that $pz = 0$ for every $z \in \bar{E}(p)$). We can now introduce the equilibrium existence theorem.

Proposition 11.19. (*Competitive equilibrium existence in pure exchange economy without free disposal*) There exists $p^* \in S$ for which $0 \in \bar{E}(p^*)$ if the aggregate excess demand correspondence $\bar{E} : S \to Z$, where Z is a compact subset of $\mathbb{R}^k$, is upper hemicontinuous, homogenous of degree zero and such that the set $\bar{E}(p)$ is non-empty, convex and satisfies Walras law for every $p \in S$.

Proof. We apply the theorem specified above because the correspondence $\bar{E} : S \to Z$ satisfies its assumptions. We find out that the second of the three alternatives suggested by the theorem is excluded while both of the other two imply $0 \in \bar{E}(p^*)$. The first alternative indicates $0 \in \bar{E}(p^*)$. The second alternative cannot occur because Walras law requires $p^*z = 0$ for every $z \in \bar{E}(p^*)$, while it requires $p^* z^* = \|z^*\| \neq 0$ since $z^* \neq 0$. The condition $\dfrac{\hat{z}}{\|\hat{z}\|} = -\dfrac{z^*}{\|z^*\|}$ required by the third alternative

endowments allowed all the agents to get their satiation consumption, a case that can be referred to as a paradise economy and not a real one.

[14] This theorem is by Hart and Kuhn, 1975, p. 336.

can be represented by $\lambda z^* + (1-\lambda)\hat{z} = 0$ with $\lambda = \frac{\|\hat{z}\|}{\|z^*\| + \|\hat{z}\|}$. Since the correspondence $\bar{E} : S \to Z$ is homogenous of degree zero we get that $z^*, \hat{z} \in \bar{E}(p^*)$ because $\bar{E}(p^*) = \bar{E}(-p^*)$. Then, since $\bar{E}(p^*)$ is a convex set and $z^*, \hat{z} \in \bar{E}(p^*)$, we get that $\lambda z^* + (1-\lambda)\hat{z} \in \bar{E}(p^*)$ for every $\lambda \in (0, 1)$, therefore also for $\lambda = \frac{\|\hat{z}\|}{\|z^*\| + \|\hat{z}\|}$, so as a result $0 \in \bar{E}(p^*)$. □

As the first remark, note that if (x^*, p^*) is a competitive equilibrium, then also $(x^*, -p^*)$ is a competitive equilibrium. They are, nevertheless, only apparently different equilibria: the vectors of accounting prices p^* and $-p^*$ induce the same exchange ratios.[15]

The more relevant remark considers equilibrium efficiency. Contrary to the case of the economies with free disposal, the first welfare theorem requires the introduction of a new assumption on the preferences of the agents (which is implicitly satisfied in the economies with free disposal[16]).

[15] Here, we show an economy for which there exists a "weak" equilibrium while there does not exist a "strong" equilibrium (this distinction was explained in footnote 12). Let's consider an economy with two consumers and two goods, defined by the consumption sets $X_1 = X_2 = [0, 2]^2$, the endowments $\omega_1 = \omega_2 = (1, 1)$ and the utility functions $u_1 = \begin{cases} -x_{11}^2 - x_{12}^2 & \text{per } x_{11} + x_{12} \le 2 \\ -(x_{11}-2)^2 - (x_{12}-2)^2 & \text{per } x_{11} + x_{12} \ge 2 \end{cases}$ and $u_2 = -|x_{21} + x_{22} - 2|$. This economy, also if the preferences are not weakly convex, has a unique "weak" equilibrium $x_1^* = x_2^* = (1, 1)$, $p^* = \pm(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}})$, that is also strongly efficient. It does not have any "strong" equilibrium. Instead, it has multiple equilibria with free disposal (any prices in the simplex give rise to equilibrium). In Figure 11.16 and 11.17 we depict the indifference curves maps of the consumers, on which it is easy to show the "weak" competitive equilibrium.

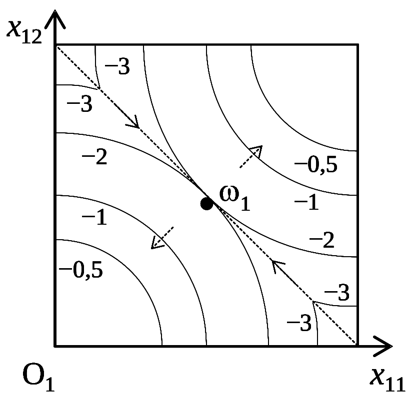

Figure 11.16

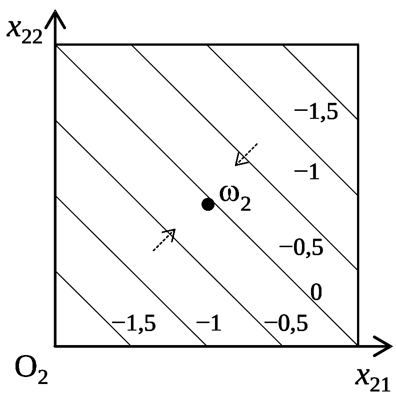

Figure 11.17

[16] And also in the analysis of "strong" equilibrium.

Preferences have to exhibit a common element: either all agents like an increase of nominal wealth or all agents like a decrease of nominal wealth.

In Figure 11.18 we show, in Edgeworth-Pareto box diagram, an economy with a unique competitive equilibrium that is inefficient (this economy, with two consumers and two goods is characterized by consumption sets $X_1 = X_2 = [0,\ 6]^2$, endowments $\omega_1 = (2, 2)$, $\omega_2 = (4, 4)$ and utility functions $u_1 = -(x_{11} - 1)^2 - (x_{12} - 1)^2$, $u_2 = x_{21} + x_{22}$. The equilibrium is $x_1^* = (2, 2)$, $x_2^* = (4, 4)$, $p^* = \pm(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}})$).[17]

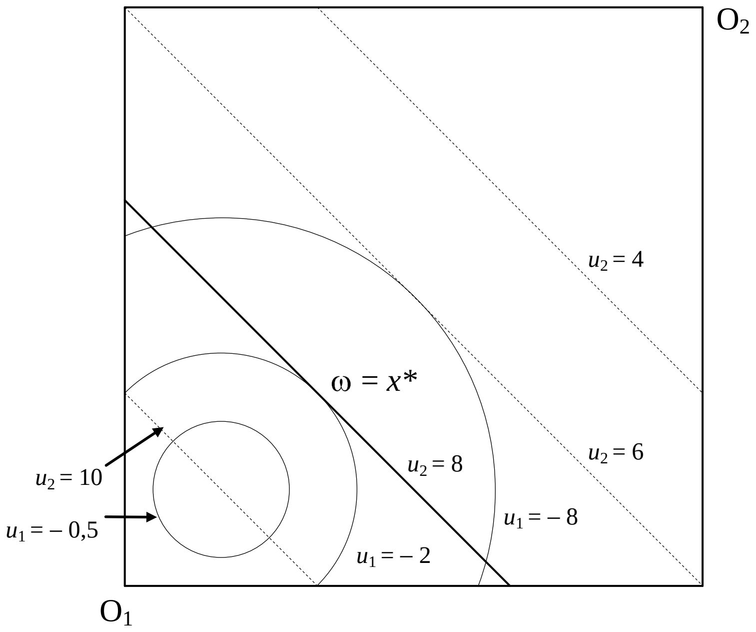

Figure 11.18

The proof of the first welfare theorem for the case without free disposal requires the following assumption.

Definition 11.6 (*Dislike for a decrease of nominal wealth*) The *i*-th consumer weakly dislikes a decrease of nominal wealth with respect to (p, ω_i) if $x_i' \precsim_i \overline{d}_i(p)$ (that is $x_i' \precsim_i x_i$ for every $x_i \in \overline{d}_i(p)$) for every $x_i' \in X_i$ with $px_i' < p\omega_i$. He strongly dislikes if $x_i' \prec_i \overline{d}_i(p)$. He weakly (strongly)

[17] This "weak" equilibrium is not "strong", so this economy does not have any strong equilibrium. If we allow for free disposal, then we find out that there exists the unique equilibrium $x_1^* = (1, 1)$, $x_2^* = (4, 4)$, $p^* = \pm(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}})$, that is weakly (but not strongly) efficient.

dislikes an increase of nominal wealth if $x_i' \precsim_i \overline{d}_i(p)$ ($x_i' \prec_i \overline{d}_i(p)$) for every $x_i' \in X_i$ with $px_i' > p\omega_i$.[18]

Proposition 11.20 The *i*-th consumer weakly dislikes, with respect to any pair (p, ω_i), where ω_i is a point in the interior of X_i, either a decrease or an increase of nominal wealth if the consumption set X_i is non-empty, closed and convex and the system of preferences $\langle X_i, \succsim_i \rangle$ is regular, continuous and weakly convex. (In other words, with these assumptions, it must be that $x_i' \precsim_i \overline{d}_i(p)$ for every $x_i' \in X_i$ with $px_i' < p\omega_i$ or for every $x_i' \in X_i$ with $px_i' > p\omega_i$). He strongly dislikes if the preferences are locally nonsatiated (as defined in Paragraph 3.2).

Proof. Let's consider a quadruple (p, ω_i, x_i', x_i'') where $p \in S$, ω_i is a point in the interior of X_i and $x_i', x_i'' \in X_i$ with $px_i' < p\omega_i$ and $px_i'' > p\omega_i$. The proposition is proved if we show that it is impossible to have $x_i' \succ_i \overline{d}_i(p)$ and $x_i'' \succ_i \overline{d}_i(p)$ at the same time. Let $x_i^* = \lambda x_i' + (1-\lambda)x_i''$ with $\lambda = \dfrac{p(x_i'' - \omega_i)}{p(x_i'' - x_i')}$ so that $px_i^* = p\omega_i$. Also $x_i^* \in X_i$ since X_i is convex and $\lambda \in (0, 1)$. Therefore, $x_i^* \precsim_i \overline{d}_i(p)$. Then, if $x_i' \succsim_i x_i''$, since the preferences are weakly convex, we get $x_i^* \succsim_i x_i''$, by which (since $x_i^* \precsim_i \overline{d}_i(p)$) we get $x_i'' \precsim_i \overline{d}_i(p)$; if $x_i' \precsim_i x_i''$, then also $x_i^* \succsim_i x_i'$, by which $x_i' \precsim_i \overline{d}_i(p)$. Since at least one of the following $x_i'' \precsim_i \overline{d}_i(p)$ or $x_i' \precsim_i \overline{d}_i(p)$ must hold, then $x_i' \succ_i \overline{d}_i(p)$ and/or $x_i'' \succ_i \overline{d}_i(p)$ are false.. We will now show that it is impossible that both $x_i' \succsim_i \overline{d}_i(p)$ and $x_i'' \succsim_i \overline{d}_i(p)$ are true if the preferences are locally nonsatiated. This assumption implies that there is a point $\tilde{x}_i'$ in the neighborhood of x_i' such that $p\tilde{x}_i' < p\omega_i$ and $\tilde{x}_i' \succ_i x_i'$, and, in the neighborhood of x_i'', a point $\tilde{x}_i''$ such that $p\tilde{x}_i'' > p\omega_i$ and $\tilde{x}_i'' \succ_i x_i''$. Proceeding as in the preceding part of the proof, that is introducing a point $\tilde{x}_i^* = \tilde{\lambda}\tilde{x}_i' + (1-\tilde{\lambda})\tilde{x}_i''$ with $\tilde{\lambda} = \dfrac{p(\tilde{x}_i'' - \omega_i)}{p(\tilde{x}_i'' - \tilde{x}_i')}$, we find that if $\tilde{x}_i' \succsim_i \tilde{x}_i''$, then also $\tilde{x}_i^* \succsim_i \tilde{x}_i''$, by which (since $\tilde{x}_i^* \precsim_i \overline{d}_i(p)$) we get that $x_i'' \prec_i \tilde{x}_i'' \precsim_i \overline{d}_i(p)$; if $\tilde{x}_i'' \succsim_i \tilde{x}_i'$, then also $\tilde{x}_i^* \succsim_i \tilde{x}_i'$, by which we get $x_i' \prec_i \tilde{x}_i' \precsim_i \overline{d}_i(p)$. As a consequence, at least one of the two relationships $x_i'' \prec_i \overline{d}_i(p)$ and $x_i' \prec_i \overline{d}_i(p)$ must hold. □

[18] Note that a consumer always weakly dislikes a decrease of the nominal wealth if his choice is represented by the demand function $d_i(p)$ (which is taken under consideration by the "strong" equilibrium analysis, in which $p > 0$). He strongly dislikes a decrease if the preferences are locally nonsatiated.

We can now introduce the first welfare theorem for a pure exchange economy without free disposal.

Proposition 11.21 (*First Welfare Theorem*) A competitive equilibrium (x^*, p^*) without free disposal of an economy $\mathcal{E} = (\langle X_i, \succsim_i \rangle, \omega_i, i = 1,\dots, n)$ is weakly (strongly) efficient if all consumers (strongly) dislike a decrease or an increase of nominal wealth with respect to (p^*, ω_i). (In other words, we require that there is similarity between consumers, in the sense that they all dislike a decrease of nominal wealth or that they all dislike an increase of nominal wealth).

Proof. Let's consider an equivalent proposition according to which if an allocation is not weakly efficient, then it cannot belong to a competitive equilibrium with prices p^*. If x is not a weakly efficient allocation, then there exists a feasible allocation x', that is with $\Sigma_{i=1}^n x_i' = \Sigma_{i=1}^n \omega_i$ and so $\Sigma_{i=1}^n p^* x_i' = \Sigma_{i=1}^n p^* \omega_i$, such that $x_i' \succ_i x_i$ for every $i = 1,...,n$. If $p^* x_i' = p^* \omega_i$ for some i, then $x_i \notin \bar{d}_i(p^*)$, as $x_i' \succ_i x_i$ and $x_i' \in \bar{B}_i(p^*)$. Therefore (x, p^*) can be a competitive equilibrium only if $p^* x_i' \neq p^* \omega_i$ for every $i = 1,...,n$. However, if $p^* x_i' \neq p^* \omega_i$ for every $i = 1,...,n$, since $\Sigma_{i=1}^n p^* x_i' = \Sigma_{i=1}^n p^* \omega_i$, then there is at least one (j-th) consumer for whom $p^* x_j' > p^* \omega_j$ and one (h-th) consumer for whom $p^* x_h' < p^* \omega_h$. Since every consumer weakly dislikes the decrease (or increase) of nominal wealth, at least one of the following relationships must be true $x_j' \precsim_j \bar{d}_j(p^*)$ and/or $x_h' \precsim_h \bar{d}_h(p^*)$. Therefore, since $x_j \prec_j x_j'$ and $x_h \prec_h x_h'$, at least one of the following relationships $x_j \prec_j \bar{d}_j(p^*)$ and $x_h \prec_h \bar{d}_h(p^*)$ must hold. As a result, (x, p^*) is not a competitive equilibrium. The proof of strong efficiency is analogous. If x is not a strongly efficient allocation, then there exists a feasible allocation x', that is with $\Sigma_{i=1}^n x_i' = \Sigma_{i=1}^n \omega_i$ and so $\Sigma_{i=1}^n p^* x_i' = \Sigma_{i=1}^n p^* \omega_i$, such that $x_i' \succsim_i x_i$ for every $i = 1,...,n$ and $x_i' \succ_i x_i$ for at least one i. If $p^* x_i' = p^* \omega_i$ for every $i = 1,...,n$, so it is also for the consumer for whom $x_i' \succ_i x_i$. Then for this consumer $x_i \notin \bar{d}_i(p^*)$ since $x_i' \succ_i x_i$ and $x_i' \in \bar{B}_i(p^*)$. Therefore, (x, p^*) can be a competitive equilibrium only when it is not true that $p^* x_i' = p^* \omega_i$ for every $i = 1,...,n$. Then, since $\Sigma_{i=1}^n p^* x_i' = \Sigma_{i=1}^n p^* \omega_i$, there is at least one (j-th) consumer for whom $p^* x_j' > p^* \omega_j$ and another (h-th) consumer for whom $p^* x_h' < p^* \omega_h$. Since every consumer strongly dislikes the decrease (or increase) of nominal wealth, at least one of the following relationships must be true $x_j' \prec_j \bar{d}_j(p^*)$ and/or $x_h' \prec_h \bar{d}_h(p^*)$. Therefore, since $x_j \precsim_j x_j'$ and $x_h \precsim_h x_h'$, at least one of the following relationships $x_j \prec_j \bar{d}_j(p^*)$ and $x_h \prec_h \bar{d}_h(p^*)$ must hold. As a consequence, (x, p^*) is not a competitive equilibrium. □

The assumption used in Proposition 11.21 is a sufficient condition in order for the first welfare theorem to hold, but it is not a necessary condition. In fact, it is possible to find examples in which a competitive equilibrium is efficient even when not all the consumers weakly dislike a decrease (or increase) of nominal wealth.[19]

There is no simple relation between the *free disposal* and *non free disposal* equilibria.[20] There are economies that have a non free disposal equilibrium (also strongly efficient) and no free disposal equilibrium. There are also economies that have a free disposal equilibrium (also strongly efficient) and no non free disposal equilibrium.

In Figures 11.24 and 11.25 we show Edgeworth-Pareto box diagrams of two economies with two consumers and two goods. The indifference curves of the two consumers of the economy shown in Figure 11.24 are represented, respectively, in Figures 11.22 and 11.23 (the numbers near the indifference curves show utility). The two consumers have consumption set $X_i = [0, 4]^2$ and endowment $\omega_i = (2, 2)$ for $i = 1, 2$. There is a non free disposal competitive equilibrium that is represented by $x_i^* = \omega_i$ for $i = 1, 2$ and $p^* = \pm(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}})$. There is no free disposal equilibrium, as we see from the price-consumption curves, drawn (for the free disposal case) as dashed lines in Figures 11.22 and 11.23 and reported in Edgeworth-Pareto box diagram in Figure 11.24. In fact, in Figure 11.24, none of the points on the price-consumption curve of consumer 1 is on south-west from any point on the price-consumption curve of consumer 2, as it is necessary in order to have a free-disposal equilibrium.

[19] For an economy with two agents and two goods, we depict the preferences and endowments of the consumers (Figures 11.19 and 11.20) and the Edgeworth-Pareto box diagram (Figure 11,21). We show a competitive equilibrium that is strongly efficient even if one of the consumers strongly dislikes a decrease of nominal wealth and the other one an increase.

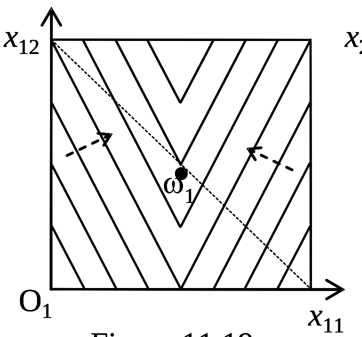

Figure 11.19

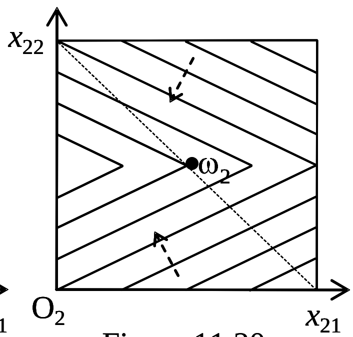

Figure 11.20

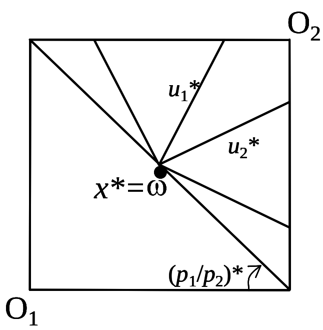

Figure 11.21

[20] On the contrary, as we have already shown, a "strong" equilibrium is also a "weak" equilibrium. Moreover, a "strong" equilibrium is weakly efficient while a "weak" equilibrium may be inefficient as shown in Figure 11.18. Nevertheless, it is also possible that a "weak" equilibrium that is also strongly efficient exists for an economy that has no "strong" equilibria, as in the case represented in Figures 11.16 and 11.17.

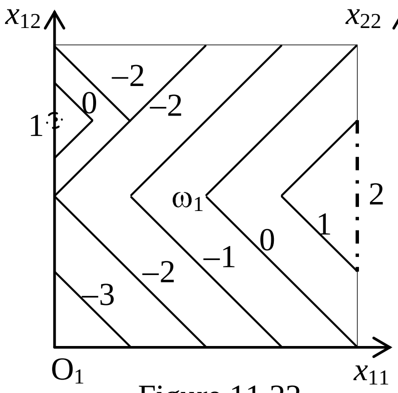

Figure 11.22

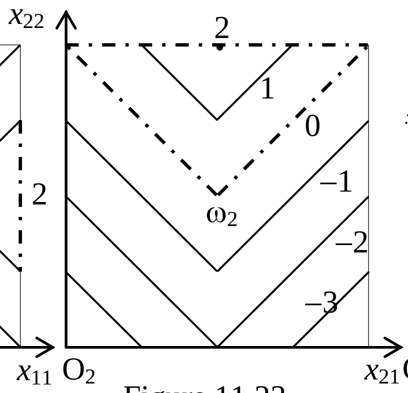

Figure 11.23

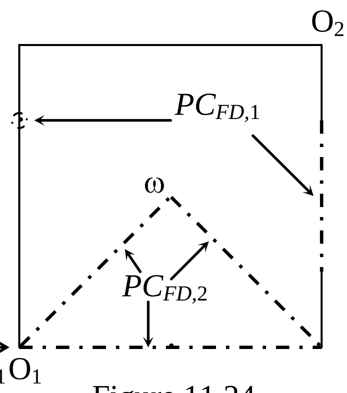

Figure 11.24

In the economy represented in Figure 11.25, the first consumer has preferences described by the indifference curves in figure and the second consumer by the utility function $u_2 = \min\{x_{21}, x_{22}\}$, with X_i = [0, 4]^2 and ω_i = (2, 2) for i=1, 2. There is the free disposal equilibrium x_1* = (0, 0), x_2* = (2, 2) and (p_1/p_2)* = 1. There is no equilibrium without free disposal as highlighted by the corresponding price-consumption curves.

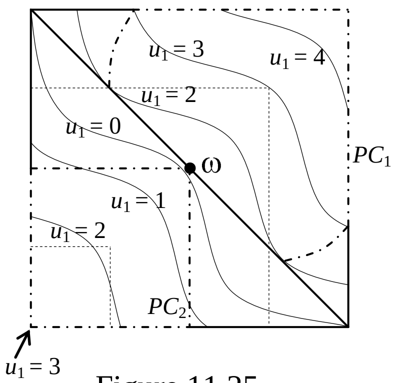

Figure 11.25

We note that the assumption that the preferences are locally nonsatiated is the crucial element in order for the competitive equilibrium to be strongly efficient, both in the case with or without free disposal.[21]

The second welfare theorem and the consideration of production for an economy without free disposal do not give any additional insight with respect to the analysis of an economy with free disposal.[22]

[21] In this last case, both for "strong" and "weak" equilibrium.
[22] For this last analyses, see Montesano, 2001.

11.13 Competitive equilibrium with a continuum of agents

The assumption that aggregate excess demand functions are continuous with respect to prices (or, in general, that excess demand correspondences are convex valued so they associate a convex set of consumption vectors to every price vector and they are upper hemicontinuous) is crucial to prove equilibrium existence (as shown in paragraphs 11.4 and 11.8). In order to ensure that this assumption is satisfied it is usually assumed (for example in paragraph 11.8) that consumers' preferences and production sets are convex. In fact, when preferences are not convex then individual choice does not, in general, satisfy the above indicated continuity condition and aggregate demand does not satisfy it as well. The same holds for supply when production sets are not convex. Nevertheless, assumption that preferences and production sets are convex is rather strong. Many consumers have preferences that are not convex. For example, a consumer can prefer a glass of red wine to a half glass of red and a half glass of white wine, and at the same time prefer a glass of white wine to a half glass of red and a half glass of white wine. The same can occur for production: for example production sets are not convex if the returns to scale are not non increasing.

Therefore, we will now study whether it is possible to avoid convexity assumption on preferences. We must accept that individual demand does not have the desired feature (it is not a continuous and/or a convex valued function and/or an upper hemicontinuous correspondence). However, these features are satisfied for aggregate demand. Such a result can be obtained if we consider a continuum, instead of a discrete number, of consumers, for example with $i \in 0,1$ instead of $i \in 1,2,...,n$. (The continuity of consumers does not imply that the number of consumers is infinite. We are in an analogous situation if we consider a continuous distribution of incomes represented by a density function of frequency $n(y)$ with $y \in y_{\min}, y_{\max}$. This does not imply infinite population: the size of population is $N = \int_{y_{\min}}^{y_{\max}} n(y)dy$ and total income is $Y = \int_{y_{\min}}^{y_{\max}} yn(y)dy$). By analogy for the individual and aggregate supply of a firm.

Example 11.1 We consider pure exchange economy where every consumer is endowed with $\omega = (\bar{m}, \bar{x})$ and (quasilinear) utility function

$$\begin{cases} u = m + \ln x & \text{for } x \le 1 \\ u = m + 1 + \ln(x-1) & \text{for } x \ge 2 \\ u = m + (x-1) - (x-1)^2 + 2(x-1)^3 - (x-1)^4 & \text{for } x \in (1,2) \end{cases}$$

where $x \in \mathbb{R}_+$ indicates consumption of a good and $m \in \mathbb{R}_+$ is the expenditure on other goods. As a result, individual demand function is

$$x = p^{-1} \text{ for } p > 1, \quad x = 1 + p^{-1} \text{ for } p < 1, \quad x \in 1,2 \text{ for } p = 1$$

where p indicates the price of the examined good. This function is discontinuous at $p = 1$ (where it is a correspondence with two isolated values). The aggregate demand, if n is an integer number, is also discontinuous for $p=1$. Competitive equilibrium requires that the following condition is satisfied: $nx = n\bar{x}$ for $p \neq 1$ and $n_1 + 2n_2 = (n_1 + n_2)\bar{x}$ for $p=1$, where n_1 is the number of consumers with $x=1$ and n_2 with $x=2$. In the discrete case, that is when n, n_1 and n_2 are integer numbers, competitive equilibrium can either exist or not: if $n=2$ and $\bar{x}=7/4$ there is no equilibrium; if $n=4$ and $\bar{x}=7/4$ there is one equilibrium with $p=1$, $n_1=1$ and $n_2=3$; if $\bar{x}=\sqrt{2}$ equilibrium does not exist independent of the number of consumers. In the continuous case, the equilibrium always exists and is represented: when $\bar{x} \in \mathbb{R}_+ \setminus (1,2)$, by $x=\bar{x}$ with $p=\frac{1}{\bar{x}}$ if $\bar{x} \in [0,1]$ and $p=\frac{1}{\bar{x}-1}$ if $\bar{x} \in [2,\infty]$; and when $\bar{x} \in [1,2]$, by $p=1$ with $x=1$ for share $\alpha_1 = 2-\bar{x}$ of consumers and $x=2$ for share $\alpha_2 = 1-\alpha_1 = \bar{x}-1$. In fact in the continuous case, the shares α_1 and α_2 are any real number in the interval [0,1], while in the discrete case they have to be rational numbers, moreover equal to the ratio between a (non negative and not higher than n) integer number and number of consumers n. In Figure 11.26 we represent individual demand function; in Figure 11.27 there are aggregate demand and supply functions for $n=2$ and $\bar{x}=7/4$; and in Figure 11.28 demand and supply functions for the case with a continuum of consumers.

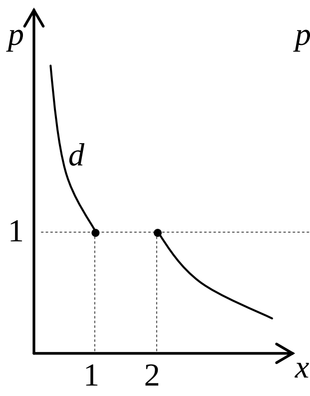

Figure 11.26

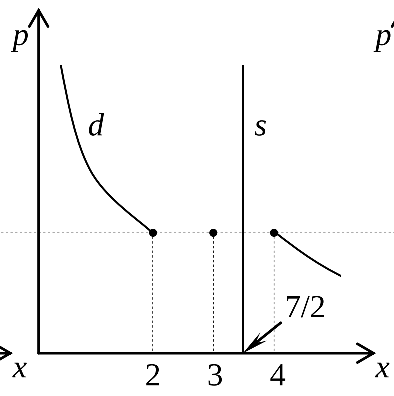

Figure 11.27

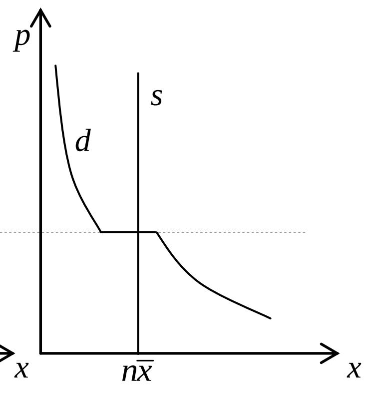

Figure 11.28

Analysis of general equilibrium with a continuum of agents was introduced by Aumann (1964 and 1966). Denoting with I the set of consumers, pure exchange economy (that is, a type of economy described in paragraph 11.3) becomes $\mathcal{E} = (\langle X(i), \succsim_i \rangle, \omega(i), i \in I)$. It differs in comparison to the discrete case only because we have $i \in I$ instead of $i \in 1,\dots,n$ (and a change in notation that introduces i as an argument of a function rather than an index). Individual consumption choice is

$$d(p;i) = \{x(i) \in B(p;i) : x(i) \succsim_i x(i)' \text{ for every } x(i)' \in B(p;i)\}$$

where $B(p;i) = \{x(i) \in X(i) : px(i) \leq p\omega(i)\}$ is a budget constraint and all the properties already stated for individual demand still hold. Feasibility condition changes and is now given by

$$\int_{i \in I} d(p;i)di = \int_{i \in I} \omega(i)di$$

(that is, in the continuous case, aggregate demand is an integral, instead of a sum, of the individual demands).

Peculiarities of the general equilibrium analysis with a continuum of agents are related to this condition, that is to the presence of an integral (instead of a sum). The crucial assumption is that the set I allows for nonatomic (Lebesgue) measure spaces. That is, it is possible to subdivide every subset of I of positive measure (with respect to the quantity of the good demanded or endowment) into two subsets with positive measure. In other words, there does not exist an $i \in I$ who has a finite endowment of goods or consumption, it can only have and consume infinitesimal quantities. With this assumption, that implies that there are no agents with market power, it is possible to prove equilibrium existence without assuming that individual preferences are convex. This assumption substitutes for the convexity of preferences when we prove that the demand correspondence is convex valued. With other usual assumptions we can obtain a proof for equilibrium existence.[23]

Indivisible goods, which quantity is represented by an integer number, are a remarkable case of non convexity (for example, cars are indivisible). In this case consumption set X_i of every consumer is non convex (not only preferences are not convex) and the individual demand correspondence has codomain composed only of integer numbers. The presence of indivisible goods can cause competitive equilibrium inexistence as shown by the following example.

Example 11.2 In a pure exchange economy there are two consumers, two indivisible goods and one infinitely divisible good, with $X_i = \{0,1\}^4 \times \mathbb{R}_+$ for $i = 1,2$. The endowments of the two consumers are respectively equal to $\omega_1 = (1,1,0,0,1)$ and $\omega_2 = (0,0,1,1,1)$ and their preferences are represented by utility functions

$$u_1 = 2(x_{11} + x_{12} + x_{13} + x_{14} + x_{11}x_{13} + x_{12}x_{14}) + x_{11}x_{12} + \frac{1}{2}x_{13}x_{14} + \frac{1}{5}(x_{11}x_{14} + x_{12}x_{13}) + \frac{4m_1}{1+m_1}$$

$$u_2 = 2(x_{21} + x_{22} + x_{23} + x_{24} + x_{21}x_{24} + x_{22}x_{23}) + x_{23}x_{24} + \frac{1}{2}x_{21}x_{22} + \frac{1}{5}(x_{21}x_{23} + x_{22}x_{24}) + \frac{4m_2}{1+m_2}$$

where $x_{ih} \in \{0,1\}$ denotes the quantity of the indivisible good, for $i = 1,2$ and $h = 1,2,3,4$, and m_i is the quantity of the divisible good for $i = 1,2$. The initial allocation (ω_1, ω_2) is Pareto efficient, with $u_1(\omega_1) = u_2(\omega_2) = 7$. In

[23] A detailed description of a model with a continuum of agents can be found in Ellickson (1993: the model by Aumann is presented in Chapter 3, pp. 99 and ss., and the problem of equilibrium existence is discussed in pp. 352-353).

fact, every other feasible allocation reduces the utility of at least one consumer: taking into account that every bundle of goods with less than two units of the indivisible goods is worse than the endowment, $u_i \le 2+\frac{8}{3} < 7 = u_i(\omega_i)$, and considering feasible allocations with two units of indivisible good per consumer, we get the following utilities for the feasible allocations

$u_1(1,1,0,0,m_1) = 5+\frac{4m_1}{1+m_1}$, $\quad u_2(0,0,1,1,m_2) = 5+\frac{4m_2}{1+m_2}$, with $m_1+m_2=2$;

$u_1(1,0,1,0,m_1) = 6+\frac{4m_1}{1+m_1}$, $\quad u_2(0,1,0,1,m_2) = \frac{21}{5}+\frac{4m_2}{1+m_2}$, with $m_1+m_2=2$;

$u_1(1,0,0,1,m_1) = \frac{21}{5}+\frac{4m_1}{1+m_1}$, $\quad u_2(0,1,1,0,m_2) = 6+\frac{4m_2}{1+m_2}$, with $m_1+m_2=2$;

$u_1(0,1,1,0,m_1) = \frac{21}{5}+\frac{4m_1}{1+m_1}$, $\quad u_2(1,0,0,1,m_2) = 6+\frac{4m_2}{1+m_2}$, with $m_1+m_2=2$;

$u_1(0,1,0,1,m_1) = 6+\frac{4m_1}{1+m_1}$, $\quad u_2(1,0,1,0,m_2) = \frac{21}{5}+\frac{4m_2}{1+m_2}$, with $m_1+m_2=2$;

$u_1(0,0,1,1,m_1) = \frac{9}{2}+\frac{4m_1}{1+m_1}$, $\quad u_2(1,1,0,0,m_2) = \frac{9}{2}+\frac{4m_2}{1+m_2}$, with $m_1+m_2=2$;

from which, as a result, we get that there are no feasible allocations better for both of the consumers than the initial endowment (because the inequalities $6+\frac{4m_i}{1+m_i} > 7$ and $\frac{21}{5}+\frac{4m_j}{1+m_j} > 7$ are incompatible for $m_i+m_j=2$). Then we conclude that the initial allocation is the only candidate for a competitive equilibrium allocation. However, it is not an equilibrium. In fact, the bundle of goods chosen by the first consumer is $\omega_1 = (1,1,0,0,1)$ only if $p_3 > p_2$ and $p_4 > p_1$ (where p_h, with $h=1,2,3,4$, are prices of the indivisible goods) since $u_1(1,0,1,0,1) > u_1(\omega_1)$ and $u_1(0,1,0,1,1) > u_1(\omega_1)$. By analogy, the bundle of goods chosen by the second consumer is $\omega_2 = (0,0,1,1,1)$ only if $p_1 > p_3$ and $p_2 > p_4$ because $u_2(1,0,0,1,1) > u_2(\omega_2)$ and $u_2(1,0,0,1,1) > u_2(\omega_2)$. However, the indicated inequalities are incompatible, the first ones requiring $p_3 + p_4 > p_2 + p_1$ and the second ones $p_1 + p_2 > p_3 + p_4$. Therefore, competitive equilibria do not exist for the examined economy (consequently, also second welfare theorem does not apply: there is one efficient allocation that is not supported by a competitive equilibrium).

Equilibrium existence proof can be obtained for the case with indivisible goods assuming that their importance is low, in the sense that there is a (sufficient) amount of divisible goods that are appreciated by consumers in such a way that an increase in their quantity in the consumption bundle is able to compensate for every decrease in the quantity of indivisible good. This way we can get an approximate equilibrium (Broome, 1972). Then the equilibrium is obtained assuming presence of a continuum of individuals and that the divisible goods are pervasively distributed (Mas-Colell, 1977), provided that there is a finite number of indivisible goods. For example, with respect to the preceding example, let there be a continuum of agents with population composed of $\alpha_1 \in [0,1]$ agents of type 1 (that is with endowments and preferences equal to those of the first consumer) and $\alpha_2 = 1 - \alpha_1$ of agents of type 2 and $\alpha_2 \geq \alpha_1$ (with $\alpha_2 \leq \alpha_1$ we get an analogous equilibrium). As a result, we get many competitive equilibria. For example there are competitive equilibria with prices $p_1 = p_2 > \frac{7}{3}$ and $p_3 = p_4 = p_1 - \frac{2}{3}$ and allocation represented by bundle of goods $(1,0,1,0,\frac{5}{3})$ for a share of consumers (all of the first type) equal to $\frac{1}{2}\alpha_1$, by bundle $(0,1,0,1,\frac{5}{3})$ for the residual share, equal to $\frac{1}{2}\alpha_1$, of the consumers of the first type, by bundle $(0,0,1,1,1)$ for share $\alpha_2 - \alpha_1$ of the consumers of the second type, by bundle $(0,1,1,0,\frac{1}{3})$ for a share equal to $\frac{1}{2}\alpha_1$ of the consumers of the second type, and by bundle $(1,0,0,1,\frac{1}{3})$ for the residual share, equal to $\frac{1}{2}\alpha_1$, of the consumers of the second type. In equilibrium consumer of the first type achieve utility $u_1 = \frac{17}{2}$ and those of the second type $u_2 = 7$.

11.14 Competitive equilibrium with a continuum of goods. Spatial economy: location and extent

The presence of continuum of agents with finite number of goods favors the possibility of competitive equilibrium existence in the sense that it yields thicker markets. In fact, as indicated, the presence of a continuum of agents allows us to relax convexity of preferences assumption . What happens if, on the other hand, we introduce a continuum of goods, leaving the number of agents finite?

We have already seen a case with a continuum of goods when examining intertemporal choice, in which (in paragraphs 6.2-3 and 4) consumption and production are introduced as continuous functions of time. Moreover, we can have a continuum of goods with respect to space (that is their location) and other qualitative elements. Competitive equilibrium existence for an economy with a continuum of goods (infinitely divisible) and with a finite number of agents does not require particular assumptions other than the usual ones (in particular convexity), from the point of view of economics. We need, however, a rather complex mathematic tool to deal with spaces with infinite dimensions. This type of analysis will not be presented here (an interested reader can refer to an excellent presentation in Mas-Colell and Zame, 1991).

A much more complex problem is represented by a case in which there is double continuity in the economy. That is, there is a continuum of goods and agents. (These economies are called *large-square economies*. See Ostroy, 1984).

Till now we considered a continuum of goods that can yet be infinitely divided. For example, considering a continuum of goods in function of the space point where they are available, the feasibility condition requires that the demanded quantity by the agents is, in any point in the space, not higher than the one available. However, the assumption of perfect divisibility cannot always be introduced. We consider, for example, the space as a good (we need space for living, for industrial factory, etc.). Now the space must be considered, in many economies, locally indivisible. It is such if we assume that every location can be used/owned by only one agent. This indivisibility is local, in the sense that a certain space can be subdivided infinitely and therefore used by a large number of agents, however with different locations. In every location only one agent uses the space. Therefore, while not all the agents can have a painting by Picasso (paintings by Picasso are indivisible goods), all can have living space (bigger or smaller, but not in the same precise location). The available space is then represented by a set (for example in physical space with three dimensions) and feasibility condition requires that the allocation is a partition of this set. That is, denoting the total space available with set $A \subset \mathbb{R}^d$ (where $d \in \{1,2,3\}$ is physical dimension of the examined space) and with 2^A its power set (that is the set of all subsets of A), a feasible allocation among n agents is represented by a vector of subsets $(E_i)_{i=1}^n$, with $E_i \in 2^A$ for $i = 1,...,n$, $E_i \bigcap E_j = \varnothing$ for every pair $i, j \in \{1,...,n\}$ with $i \neq j$, and $\bigcup_{i=1}^n E_i = A$ (or $\subseteq A$, in the case with free disposal). Let's consider a continuum of agents, with $i \in [0,1]$, each of them choosing a space in only one location, and let's denote the extent and location of the space belonging to agent *i* with the pair $(s(i), x(i))$ (so with $s(i) \in \mathbb{R}_+$ and $x(i) \in A$) and with $m(A)$ total extent of *A*. Then, the feasibility condition requires that the functions $(s(i), x(i))$ satisfy conditions $x(i) \in A$ for every $i \in [0,1]$, $x(i) \neq x(j)$ for every pair $i, j \in [0,1]$ with $i \neq j$, and $\int_{i \in [0,1]} s(i) = m(A)$ (or $\leq m(A)$).

A simple example demonstrates the complexity of the spatial competitive equilibrium analysis.

Example 11.3 Let's consider an economy in which the only good is space and the available space is one dimensional and given by an interval $A=[0,1]$. There is a continuum of agents, all with the same utility function (that has, as its object, the extent and location of the space belonging to the agent), however with different endowments. The utility function of the agents is therefore of the type $u:\mathbb{R}_+\times A\to\mathbb{R}$, so with $u=u(s,x)$, where $s\in\mathbb{R}_+$ denotes space extent and $x\in A$ the location. In the example we assume $u=-\frac{1}{s}-\ln(1+x)$. The endowment is not equal among the agents: it cannot be the same because different agents cannot be endowment with the same location. Therefore, $\omega(i)=(\sigma(i),\xi(i))$, with $i\in[0,1]$. In this example we assume $\sigma=\frac{1}{2}(1+i)^2$ and $\xi=i$. The equilibrium is represented by functions $s(i),x(i),r(x)$ that respectively denote space allocation, that is extent and location of the space for every agent, and the price of the space for every location, that is the rent distribution. In order to determine equilibrium we need to look at individual choices of the agents and introduce feasibility condition (that is peculiar for the space given the local indivisibility assumption). The choice is represented by a solution to the problem

$$\max_{(s(i),x(i))\in B(r(x);i)} u(s(i),x(i)),$$

where $B(r(x);i)=\{s(i)\in\mathbb{R}_+,x(i)\in[0,1]:s(i)r(x(i))\le\sigma(i)r(\xi(i))\}$ is the budget constraint. Introducing Lagrangian function ((understanding the dependence of the choice on i)

$$\mathrm{L}(\lambda,s,x)=u(s,x)+\lambda(\sigma r(\xi)-sr(x))$$

we obtain for the examined example (where $u=-\frac{1}{s}-\ln(1+x)$) three first order conditions

$$\sigma r(\xi)-sr(x)=0,\qquad \frac{1}{s^2}-\lambda r(x)=0,\qquad -\frac{1}{1+x}-\lambda sr'(x)=0;$$

and a second order condition

$$\begin{vmatrix} 0 & -r(x) & -sr'(x) \\ -r(x) & -2s^{-3} & -\lambda r'(x) \\ -sr'(x) & -\lambda r'(x) & \frac{1}{(1+x)^2}-\lambda sr''(x) \end{vmatrix}\ge 0$$

Then, the first order conditions require that the two following equations are satisfied

$$\sigma(i)r(\xi(i))-s(i)r(x(i))=0,\qquad r'(x(i))=-\frac{s(i)}{1+x(i)}r(x(i)),$$

and the second order condition requires (with $\lambda=\frac{1}{s^2r(x)}$)

$$-\frac{1}{(1+x(i))^2}(r(x(i))^2+\frac{1}{s(i)}r(x(i))r''(x(i))\geq 0$$

The feasibility condition of the choices implies that the demanded space extent in a given location is equal to the one therein available, and the same condition applies to the endowments. Considering the endowment, the agents of type i have space in location $\xi(i)$, where they possess extent $\sigma(i)$. Indicating with $n(i)$ the density of the agents of type i, for which $n(i)di$ is the number of agents of the type between i and $i+di$ and $\sigma(i)n(i)di$ the extent of the space that they have, we get that this extent must coincide with available extent in this location, that is $|d\xi(i)|$. Therefore we get that $\sigma(i)n(i)di=|d\xi(i)|$, that is $n(i)=\frac{1}{\sigma(i)}\left|\frac{d\xi(i)}{di}\right|$. In the example, having $\sigma=\frac{1}{2}(1+i)^2$ and $\xi=i$, we get $n(i)=\frac{2}{(1+i)^2}$. Consider now the demand. We get, by analogy, the condition $s(i)n(i)di=|dx(i)|$, by which since $n(i)=\frac{1}{\sigma(i)}\left|\frac{d\xi(i)}{di}\right|$ we get $\frac{1}{s(i)}\left|\frac{dx(i)}{di}\right|=\frac{1}{\sigma(i)}\left|\frac{d\xi(i)}{di}\right|$. In the example, given $\sigma=\frac{1}{2}(1+i)^2$ and $\xi=i$, we get condition

$$\left|\frac{dx(i)}{di}\right|=\frac{2s(i)}{(1+i)^2}.$$

Accounting for the first order conditions and assuming what was given by the example, we get the following system of three equations

$$\frac{1}{2}(1+i)^2r(i)-s(i)r(x(i))=0,$$

$$r'(x(i))=-\frac{s(i)}{1+x(i)}r(x(i)),$$

$$\left|\frac{dx(i)}{di}\right|=\frac{2s(i)}{(1+i)^2}$$

in three unknown functions $s(i),x(i),r(x)$. This differential system includes unknown functions as functions of functions. In the case of our example, this system allows for two solutions:

$$s=\frac{1}{2}(1+i)^2,\qquad x=i,\qquad r=r_0e^{-\frac{1}{4}(2x+x^2)}$$

and

$$s=1,\qquad x=\frac{1-i}{1+i},\qquad r=\frac{r_0}{1+x}.$$

However, the first solution does not satisfy second order condition. The second solution satisfies it. Therefore, there is only one competitive equilibrium, given by the second solution, with all the agents with the same space extent (that is different from the endowment), and location reversed with respect to the endowment (the agent $i = 0$ has locations $\xi = 0$ and $x = 1$, agent $i = 1$ has $\xi = 1$ and $x = 0$, and the function $\xi = i$ is increasing and $x = \frac{1-i}{1+i}$ is decreasing in *i*) and with rent distribution decreasing in *x* (as expected, because utility is decreasing with respect to location x).[24]

[24] The theme of this example belongs to the field of the spatial economics, that includes regional and urban economics. See Mills ed., 1987, and Papageorgiou and Pines, 1999. The example is of the type used in Montesano, 2003.