PhD in Statistics & Computer Science-Bocconi University
Academic Year 2026–2027

Course 40143 – Introduction to Real Analysis II

Optimal Transport

Instructor: Giuseppe Savaré

Course Description

The course provides an introduction to the theory of Optimal Transport and its applications. Optimal Transport addresses the problem of finding the best strategy to move a given distribution of masses (or goods, or physical quantities…) to a specified configuration, in order to minimise a cost depending on the initial and final positions. Starting from the pioneering work by Monge (1781) and the key contributions by Kantorovich in the early twentieth century, the subject has grown into a rich and elegant theory with deep connections to probability, statistics, convex and functional analysis, calculus of variations, partial differential equations, nonsmooth optimisation, and geometry. More recently, Optimal Transport has found remarkable applications in data science, machine learning, and the theory of generative models, where the tasks of comparing, interpolating, and dynamically evolving probability distributions plays a central role.

The aim of the course is to present the main results of Optimal Transport, supplemented by the essential tools from Measure Theory, Convex and Functional Analysis, Metric and Nonsmooth Analysis, the Area Formula, and PDE theory, with a focus on modern developments and applications.

Syllabus

Foundations and formulation

- The Monge problem: motivation, formulation, ill-posedness.
- Relaxation to the Kantorovich problem. Discrete case and linear programming.
- Review of convex analysis: Legendre-Fenchel transform, subdifferentials, Fenchel duality.
- Von Neumann minimax theorem and convex duality.
- Measure-theoretic tools: weak convergence, tightness, Prokhorov's theorem, disintegration.

Core results

- Kantorovich duality in general metrizable spaces. c-convexity, optimality conditions.
- The real line: monotone rearrangement and explicit solutions.
- The quadratic cost: Brenier's theorem, polar factorization, connections to the Monge-Ampère equation.
- The cost = distance case: Kantorovich-Rubinstein duality, W_1 and Lipschitz functions.
- Wasserstein distances: metric properties and connection to weak convergence.

Generalizations

- Entropic regularization and Schrödinger bridges. Sinkhorn's algorithm.
- Unbalanced optimal transport: Hellinger-Kantorovich distance, cone lifting, generalized duality.

Dynamic formulations and gradient flows

- The Benamou-Brenier formula: continuity equation and fluid-dynamic reformulation.
- Gradient flows in the Wasserstein space: curves of maximal slope, the JKO scheme, Fokker-Planck as a gradient flow of the free energy in W_2.

Further applications and developments *(topics may vary)*

- Wasserstein barycenters and multi-marginal optimal transport.
- Connections to Machine Learning, Optimization, and Generative Models.
- Applications to Statistics and Data Science.

Exam

The exam consists of a seminar presentation at the end of the course. The topic can be either an in-depth exploration of a subject from the syllabus or the presentation of a research paper, according to the student's interests.

References

Lecture notes and slides will be provided. The following is a selection of reference texts relating to the course topics.

Introductory and reference texts

- Alessio Figalli, Federico Glaudo, *An Invitation to Optimal Transport, Wasserstein Distances, and Gradient Flows,* EMS Press (2nd edition, 2023). doi:10.4171/etb/25
- Luigi Ambrosio, Elia Brué, Daniele Semola, *Lectures on Optimal Transport,* Springer (2021). doi:10.1007/978-3-030-72162-6
- Filippo Santambrogio, *Optimal Transport for Applied Mathematicians,* Birkhäuser (2015). doi:10.1007/978-3-319-20828-2
- Cédric Villani, *Topics in Optimal Transportation*, American Mathematical Society (2003). doi:10.1090/gsm/058
- Cédric Villani, *Optimal Transport: Old and New*, Springer (2009). doi:10.1007/978-3-540-71050-9

Computational aspects, entropic OT

- Gabriel Peyré, Marco Cuturi, *Computational Optimal Transport,* Now Foundations and Trends (2019). arXiv:1803.00567, doi:10.1561/2200000073

Wasserstein spaces and gradient flows from a metric perspective

- Luigi Ambrosio, Nicola Gigli, Giuseppe Savaré, *Gradient Flows. In Metric Spaces and in the Space of Probability Measures,* Birkhäuser (2008). doi:10.1007/978-3-7643-8722-8

Optimal transport from a statistical perspective

- Sinho Chewi, Jonathan Niles-Weed, Philippe Rigollet, *Statistical Optimal Transport* (2024). arXiv:2407.18163

Optimal transport from an economic perspective

- Alfred Galichon, *Optimal Transport Methods in Economics,* Princeton University Press (2016)